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Fidelities in the spin-boson model

Sergei L. Lukyanov[1,2]

[1]NHETC, Department of Physics and Astronomy
Rutgers University
Piscataway, NJ 08855-0849, USA

and

[2]L.D. Landau Institute for Theoretical Physics
Chernogolovka, 142432, Russia

Abstract

The spin-boson model (or the dissipative two-state system) is a model for the study of dissipation and decoherence in quantum mechanics. The spin-boson model with Ohmic dissipation is an integrable theory, related to several other integrable systems including the anisotropic Kondo and resonant level models. Here we consider the problem of computing the overlaps between two ground states corresponding to different values of parameters of the Ohmic spin-boson Hamiltonian. We argue that this can be understood as a part of the problem of quantizing the mKdV/sine-Gordon integrable hierarchy. The main objective of this work is to analyze how the Anderson orthogonality affects the Yang-Baxter integrable structure underlying the theory.

Contents

1 Introduction

The occurrence of infrared (IR) divergences is a central issue for quantum field theories and condensed matter systems which possess gapless excitations [1,2]. Boundary Conformal Fields Theories (CFT) in two-dimensional space-time provide an opportunity to gain useful insights into the problem. In the simplest set up, with the Euclidean geometry of the half-infinite plane (x, y) in which $x \leq 0$ is treated as a space coordinate, the vacuum states corresponding to different conformal Boundary Conditions (BC), say "1" and "2", are orthogonal. More precisely, for a large but finite system of space size L their overlap tends to zero as a power $L^{-d_{21}}$, defining an orthogonality exponent d_{21}. This vacuum overlap can be interpreted as a one-point function of the BC changing operator $\mathcal{O}_{21}$ [3], and therefore its contribution to the spectral sum of the two-point function $\langle\, \mathcal{O}^\dagger_{21}(y)\mathcal{O}_{21}(0)\,\rangle$ vanishes, as well as individual contributions of all overlaps involving conformal descendant states. However, the combined contribution of the states organized in conformal towers turns out to be finite for the infinitely large system and gives rise to the scale-invariant two-point function $\langle\, \mathcal{O}^\dagger_{21}(y)\mathcal{O}_{12}(0)\,\rangle = |A_{21}|^2\ |y|^{-2d_{21}}$. The latter coincides with the ratio $\mathcal{Z}_{21}(y)/\mathcal{Z}_{11}$, with $\mathcal{Z}_{21}$ standing for a partition function of the half-infinite system with BC "1" everywhere except on a part of the boundary of length y, where BC "2" is imposed, whereas the denominator $\mathcal{Z}_{11}$ is a partition function of the system with BC "1" is imposed along the whole boundary. There is generally a normalization ambiguity

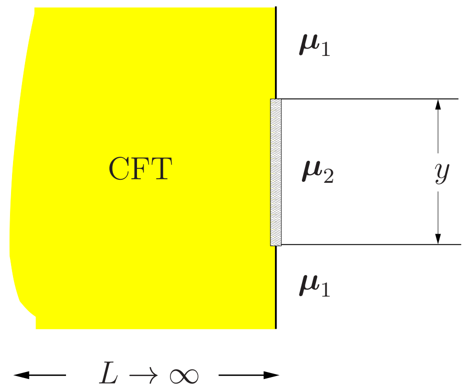

Figure 1: An universal part of the overlap modulus $|\langle\,\Omega_2\,|\,\Omega_1\,\rangle|$ can be extracted from the partition function of a classical statistical system with an inhomogeneous boundary. In this picture $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are two numerically different sets of values of boundary couplings.

of the numerical coefficient A_{21}, and usually it can be set as 1. The situation becomes more interesting for the so-called boundary flows, i.e., for a class of two-dimensional quantum field theories in which conformal invariance is broken only by BC depending upon a set of couplings $\boldsymbol{\mu} = (\mu_1, \mu_2 \ldots)$. Contrary to the conformally invariant case, the partition function of the system schematically visualized in Fig.1 is a complicated function of Euclidean time y. However, its $y \to +\infty$ asymptotic is expected to have the form

$$\frac{\mathcal{Z}_{21}(y)}{\mathcal{Z}_{11}} = |A_{21}|^2\ y^{-2d_{21}}\ \mathrm{e}^{-y\Delta E_{21}}\big(1 + o(1)\big)\ , \tag{1.1}$$

where $\Delta E_{21} \equiv E_0(\boldsymbol{\mu}_2) - E_0(\boldsymbol{\mu}_1)$ is the difference in the ground state energies for the different sets of boundary couplings. The orthogonality exponent d_{21} and the prefactor A_{21} are functions of $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ which virtually define an universal (i.e. independent on details of both IR and ultraviolet (UV) regularizations) part of the vacuum overlap

$$\langle\,\Omega_2\,|\,\Omega_1\,\rangle \propto A_{21}\ L^{-d_{21}} \qquad \text{as} \quad L \to \infty\ . \tag{1.2}$$

In common nomenclature, the modulus of this overlap is referred as to the (ground state) fidelity. Throughout this paper, with some abuse of conventional terminology, this term will be used to denote the scaling function A_{21} as well as its generalization. The generalization deals with the vacuum-vacuum matrix elements of a bare (unrenormalized) boundary field $\mathcal{O}(y)$ characterized by a certain scaling exponent $D(\mathcal{O})$, so that (1.2) is substituted by

$$\langle\,\Omega_2\,|\,\mathcal{O}(0)\,|\,\Omega_1\,\rangle \propto A_{21}\big(\mathcal{O}\big)\ \varepsilon^{D(\mathcal{O})}\ L^{-d_{21}(\mathcal{O})}\ , \tag{1.3}$$

defining both the IR exponent $d_{21}(\mathcal{O})$ and the fidelity $A_{21}\big(\mathcal{O}\big)$ (here $\varepsilon \to 0$ is the lattice spacing, i.e., the UV regulator).

The significance of the study of fidelities is that it may help to better understand universal aspects of the dynamics after a local quantum quench in quantum impurity models [6–8]. Such models are used to mimic the behavior of small interacting quantum mechanical systems coupled to an external environment. In some cases, they display universality which can be described in terms of the boundary flows (see ref. [4] for review of applications of the boundary

flows in condensed matter physics). Here we will discuss the so-called spin-boson model (or the dissipative two-state system) which is a paradigm model for study of dissipation and decoherence in quantum mechanics [5]. In the case of Ohmic dissipation, the model consists of a single two-state system coupled linearly to an infinite bath of harmonic oscillators, and described by the Hamiltonian

$$\boldsymbol{H} = \int_0^\infty \mathrm{d}k\, b_k^\dagger b_k\ \sigma_0 - J\sigma_1 - h\,\sigma_3 - \sqrt{\frac{g}{2}}\int_0^\infty \mathrm{d}k\ (b_k^\dagger + b_k)\ \sigma_3\ , \tag{1.4}$$

where the Pauli matrixes and $\sigma_0 \equiv 1$ describe the two-state system ("quantum spin"), $b_k^\dagger$ and b_k are phonon creation and annihilation operators such that $[b_k, b_{k'}^\dagger] = k\,\delta(k-k')$, $[b_k, b_{k'}] = [b_k^\dagger, b_{k'}^\dagger] = 0$. The bare tunneling amplitude between the eigenstates of σ_1 is given by J, and h is an additional bias. The Ohmic dissipative two-state system is related to several other models, including the anisotropic Kondo model [9–11], the resonant level model [12] and the inverse square Ising model [13].

One important issue in the spin-boson model is the phonon-induced delocalized-localized transition. Such a delocalized transition at zero temperature is now considered as some kind of quantum phase transition. For the Hamiltonian (1.4) the quantum transition of Kosterlitz-Thouless type occurs at $g = 1$. The delocalized region $0 < g < 1$ corresponds to the antiferromagnetic Kondo model, while the localized region corresponds to the ferromagnetic case. Here we will consider only the case $0 < g < 1$.

As it was pointed out in ref. [14] the Ohmic bath of harmonic oscillators can be interpreted as a simple bulk CFT – the massless Gaussian model. This allows one to reformulate the spin-boson model in the delocalized regime as a boundary flow problem, where the parameters of the Hamiltonian J and h play the rôle of the dimensionful boundary couplings. The flow starts from the Gaussian CFT with the Neumann (free) BC and with the decoupled spin degrees of freedom. For $h = 0$, the Gaussian field still satisfies the Neumann BC in the IR fixed point; however, the spin proves to be completely screened (for details see, e.g., ref. [4]).

In this work we will study the vacuum overlaps (1.2) where the vacuums corresponds to different sets of the couplings (J_1, h_1) and (J_2, h_2). The arguments similar to that for the X-ray edge problem [15–17] leads to the simple formula for the IR singularity exponent

$$d_{21} = \frac{g}{4}\ (m_2 - m_1)^2\ , \tag{1.5}$$

where $m_i = \langle \Omega_i\,|\sigma_3\,|\,\Omega_i\,\rangle/\langle \Omega_i\,|\,\Omega_i\,\rangle$. Despite the lack of a rigorous proof, there are strong indications, including numerical results from ref. [18], that this is an exact relation for the spin-boson model with $0 < g < 1$. The aim of this work is to make steps towards the exact calculation of fidelities.

The paper is organized as follows. In Sec. 2 we give a brief account of the basic concepts and facts and set up notations that will be used in the main body of the text. Sec. 3 reviews several well-known techniques for study of the orthogonality exponent and fidelities in the spin-boson model. The purpose of the next two sections is to develop a non-perturbative approach for a calculation of the fidelities. In the absence of IR divergences the Gell-Mann and Low theorem [19] allows one to express the vacuum overlaps in terms of the half-infinite time evolution operators in the interaction picture. However, the procedure which is based on the adiabatic switch of interaction generally fails for a system with gapless excitations. Our

approach is based on an axiomatic determination of the fidelities similar in philosophy to the form-factor bootstrap [20]. In Sec. 4 we argue that, in the case of spin-boson model, matrix components of the half-infinite time evolution operators can be interpreted as the quantum Jost operators – the quantum counterpart of the Jost functions for the pair of Sturm-Liouville equations. With this observation, the calculation of the fidelities can be considered as a part of the problem of quantizing the mKdV/sine-Gordon integrable hierarchy. The keystone element of quantum integrability is the Yang-Baxter type algebras with commutation relations defined by certain quantum R-matrix. In Sec.5, basing on the results of the works [21, 22], we propose a set of algebraic relations for the quantum Jost operators, which is then translated into a set of functional equations imposed on the fidelities. Currently, the solution of the system of functional equations is known for the case $h_1 = h_2 = 0$ only. It was reported in ref. [23]. The last section of the paper contains a few remarks concerning the fidelities $\mathcal{A}_{12}(\sigma_s^{(a)})$ corresponding to a family of bare operators

$$\big[\sigma_s^{(a)}(0)\big]_{\text{bare}} = \exp\left(a\sqrt{2g}\ \int_0^\infty \frac{\mathrm{d}k}{k}\ (b_k^\dagger - b_k)\right)\ \sigma_s\ , \tag{1.6}$$

where $\sigma_s \in \big\{\,1, \sigma_3, \frac{1}{2}\,(\sigma_1 \pm \mathrm{i}\sigma_2)\,\big\}$ and a is a real parameter. Here we also present formulas for $\mathcal{A}_{12}(\sigma_s^{(a)})$ in the case $h_1 = h_2 = 0$, which generalize the result of [23]. A derivation of these formulas is somewhat technical and it remained beyond the scope of this work.

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2 Preliminaries

2.1 Basics of Gaussian model with Neumann BC

We first consider the Gaussian model on the half-line whose dynamics is governed by the Hamiltonian,

$$H_{\text{free}} = \frac{1}{4\pi g}\ \int_{-\infty}^{0} \mathrm{d}x\ \big(\,\Pi^2 + (\partial_x\Phi)^2\,\big)\ , \tag{2.1}$$

the Neumann BC, $\partial_x\Phi(x,t)|_{x=0} = 0$, and the canonical commutation relations $[\,\Phi(x),\,\Pi(x')\,] = 2\pi\mathrm{i}\,g\ \delta(x-x')$, *etc.* The space of states splits up into the Fock spaces $\mathcal{F}_p$ – irreps of the algebra of creation-annihilation operators

$$b_k = \sqrt{\frac{2}{g}}\int_{-\infty}^{0}\frac{\mathrm{d}x}{4\pi}\ \big((\Pi+\partial_x\Phi)\,\mathrm{e}^{\mathrm{i}kx} + (\Pi-\partial_x\Phi)\,\mathrm{e}^{-\mathrm{i}kx}\big)\ \ :\qquad [b_k, b_{k'}] = k\ \delta(k+k')\,, \tag{2.2}$$

whose highest weight vectors are defined by the conditions $b_k\,|\,p\,\rangle = 0\ (k > 0)$ and $b_0\,|\,p\,\rangle = \sqrt{2g}\,p\,|\,p\,\rangle$. Throughout this paper, we will refer to $|\,p\,\rangle$ as *p-vacuums.* The Fock spaces are naturally equipped with the inner product defined by the conjugation $b_k^\dagger = b_{-k}$ and $\langle\,p'\,|\,p\,\rangle = \delta_{p',p}$.

Since $H_{\text{free}} = \int_0^\infty \mathrm{d}k\ b_{-k}b_k$, the Hamiltonian acts invariantly on each Fock space $\mathcal{F}_p$. Furthermore, all the p-vacuums correspond to the same zero-point energy, so that the ground state of the Gaussian theory with Neumann BC is a certain linear combination of the p-vacuums. With

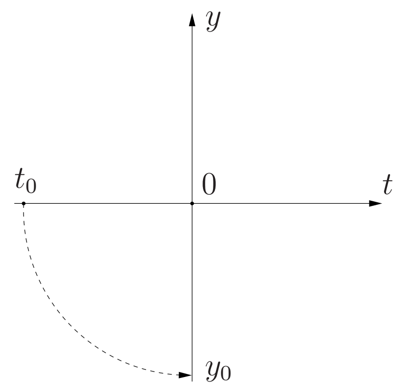

Figure 2: The Wick rotation. In the Euclidean picture, the model can be interpreted as a two-dimensional classical statistical system on the half-plane $\Re e(z) \leq 0$ with $z = x + \mathrm{i}y$.

the aim to define the ground state unambiguously, it is useful to consider the problem in the Euclidean picture where t is replaced by the Euclidean time via the Wick rotation $t \mapsto y = \mathrm{e}^{\frac{\mathrm{i}\pi}{2}}\, t$ (see Fig. 2). In the Euclidean picture all the fields are treated as functions of (x, y). Then, the ground state can be defined through the asymptotic condition

$$\lim_{y \to -\infty} \mathrm{e}^{\mathrm{i}a\Phi(x,y)} \mid \mathrm{vac} \rangle = \mid \mathrm{vac} \rangle \ , \tag{2.3}$$

which holds true for an arbitrary real parameter a. The exponential fields act between the Fock spaces, $\mathrm{e}^{\mathrm{i}a\Phi(x,y)} : \mathcal{F}_p \mapsto \mathcal{F}_{p+a}$, and we will always assume the normalization condition $\langle\, p + a \mid \mathrm{e}^{\mathrm{i}a\Phi(x,y)} \mid p\, \rangle = 1$. Thus the ground state can be written in the form of a direct integral

$$\mid \mathrm{vac} \rangle = \int_{-\infty}^{\infty} \mathrm{d}p \ \mid p \,\rangle \ . \tag{2.4}$$

The Hilbert space of the model is given by a linear span

$$\mathcal{H} = span\big\{\, b_{-k_1} \ldots b_{-k_N} \mid \mathrm{vac} \rangle \mid k_i > 0 \ \& \ N = 0, 1, \ldots \,\big\} \ . \tag{2.5}$$

Note that exponentials $\mathrm{e}^{\mathrm{i}a\Phi(x,y)}$ act invariantly on $\mathcal{H}$.

The Gaussian model is manifestly invariant under the transformation $\Phi(x, y) \mapsto -\Phi(x, y)$, which will be referred below to as C-conjugation. The corresponding symmetry operator acts as

$$\mathbb{C} \ : \qquad \mathbb{C}\, b_k = -b_k\, \mathbb{C} \ , \qquad \mathbb{C} \mid \mathrm{vac} \rangle = \mid \mathrm{vac} \rangle \ . \tag{2.6}$$

Another global symmetry is the T-invariance. The *antiunitary* T-transformation acts according to the rule $\Phi(x, y) \mapsto \Phi(x, -y)$ and the corresponding symmetry operator $\mathbb{T}$ can be defined by the relations

$$\mathbb{T} \ : \qquad \mathbb{T}\, b_k\, \mathbb{T} = -b_{-k} \ , \qquad \mathbb{T} \mid \mathrm{vac} \rangle = \mid \mathrm{vac} \rangle \ . \tag{2.7}$$

Finally let us note that the Gaussian field $\Phi(x, y)$ splits into holomorphic and antiholomorphic components

$$\Phi(x, y) = \phi(x + \mathrm{i}y) + \phi(-x + \mathrm{i}y) \ . \tag{2.8}$$

In fact, the Gaussian CFT with Neumann BC can be interpreted as a model of a chiral bose field on the "unfolded" half-infinite line, whose Euclidean time evolution, $\phi(x,y)=\phi(x+\mathrm{i}y)$, is produced by the Hamiltonian

$$H_{\rm free}=\frac{1}{2\pi g}\ \int_{-\infty}^{+\infty}\mathrm{d}x\,\big(\partial_x\phi\big)^2 \tag{2.9}$$

through the commutation relation

$$[\phi(x_2),\phi(x_1)]=\frac{\mathrm{i}}{2}\,\pi g\ \mathrm{sgn}(x_2-x_1)\ . \tag{2.10}$$

2.2 Renormalization in the spin-boson model

We now turn to the model of boundary interaction with the Hamiltonian

$$\boldsymbol{H}=H_{\rm free}\,\sigma_0-\big(\tfrac{1}{2}\,\Pi_B+h\big)\,\sigma_3-J\,\sigma_1\ , \tag{2.11}$$

where $H_{\rm free}$ has been defined by eq.(2.1) and $\Pi_B\equiv\Pi(x)|_{x=0}$. This Hamiltonian acts in the tensor product of the Hilbert space (2.5) and the two-dimensional linear space whose endomorphisms spanned by the conventional 2×2 Pauli matrices and $\sigma_0\equiv\begin{pmatrix}1&0\\0&1\end{pmatrix}$. The Hamiltonian is hermitian for real values of the parameters J and h. Without loss of generality one can assume that $J\geq 0$. In terms of the creation-annihilation operators $b_k=b^\dagger_{-k}$ (2.2) the Hamiltonian $\boldsymbol{H}$ is given by eq.(1.4). In this form it occurs as a particular realization of the Caldeira-Leggett Hamiltonian [5]. The model is usually referred to as the spin-boson model or dissipative two-level system and used to mimic the behavior of dissipative particle confined in a double-well potential.

The spin-boson model needs renormalization. The Hamiltonian (1.4) has to be equipped with the ultraviolet cut-off Λ and consistent removal of the UV divergences requires the bare coupling constants J and g be given a dependence of the cut-off according to Renormalization Group (RG) flow equations. There exists a RG scheme where

$$\Lambda\,\frac{\mathrm{d}J}{\mathrm{d}\Lambda}=g\ J\ ,\qquad \Lambda\,\frac{\mathrm{d}g}{\mathrm{d}\Lambda}=0\ , \tag{2.12}$$

and because of this one can substitute the bare coupling J by the RG invariant energy scale

$$E^\star=const\ \Lambda\,\Big(\frac{J}{\Lambda}\Big)^{\frac{1}{1-g}}\ . \tag{2.13}$$

The latter is defined up to a multiplicative g-dependent constant and usually referred to as Kondo temperature in the context of the anisotropic Kondo model. The parameter h is interpreted as an external magnetic field applied to the impurity spin. It is often convenient to specify the Kondo temperature as

$$E^\star=-\bigg[\frac{\partial^2}{\partial h^2}E_0(J,h)\bigg]^{-1}_{h=0}\ , \tag{2.14}$$

where E_0 stands for the ground state energy considered as a function of the bare coupling J and h.

Perhaps the simplest way to understand the renormalization scheme (2.12) is based on an alternative form of the Hamiltonian $\boldsymbol{H}$ (2.11). As it was already mentioned, the Gaussian theory with Neumann BC can be interpreted as a model of a free chiral boson. Eq.(2.8) implies that $\Pi_B = -2\,\partial_x\phi(0)$ and, therefore, the canonical transformation $\boldsymbol{H} \mapsto \boldsymbol{U}^\dagger \boldsymbol{H} \boldsymbol{U}$ with $\boldsymbol{U} = \exp\big(\mathrm{i}\sigma_3\phi(0)\big)$ brings the Hamiltonian (2.11) to the form

$$\boldsymbol{H}_\phi = \boldsymbol{U}^\dagger \boldsymbol{H} \boldsymbol{U} = \frac{1}{2\pi g}\int_{-\infty}^{+\infty} \mathrm{d}x\,\big(\partial_x\phi\big)^2\,\sigma_0 - h\,\sigma_3 - \mu\,\big(\mathrm{e}^{+2\mathrm{i}\phi(0)}\,\sigma_- + \mathrm{e}^{-2\mathrm{i}\phi(0)}\,\sigma_+\big) \tag{2.15}$$

with $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm \mathrm{i}\sigma_2)$ and $[\phi(x), \phi(x')] = \frac{\mathrm{i}}{2}\,\pi g\,\mathrm{sgn}(x-x')$. Notice that in eq.(2.15) it is assumed that the Hamiltonian is expressed in terms of the *renormalized* exponential operators so that the bare coupling $J = J(\Lambda)$ is substituted by the RG-invariant μ. In order to assign a precise meaning to the renormalized coupling one needs to specify a normalization condition for the renormalized exponentials. In fact, we have already accept the condition $\langle\, p+a\,|\,\mathrm{e}^{2\mathrm{i}a\phi(y)}\,|\,p\,\rangle = 1$. This sets a value of the leading term of the Euclidean operator product expansion

$$\mathrm{e}^{2\mathrm{i}\phi(y)}\,\mathrm{e}^{-2\mathrm{i}\phi(-y)} \to\ (+1)\times(2y)^{-2g} \qquad \text{as}\ \ y = \mathrm{i}t \to 0 \qquad (y>0)\ , \tag{2.16}$$

where $\mathrm{e}^{-2\mathrm{i}\phi(-y)} = \big(\mathrm{e}^{2\mathrm{i}\phi(y)}\big)^\dagger$. Thus the renormalized coupling μ has the dimension of $[\,\mathit{energy}\,]^{1-g}$, i.e., $\mu = J\Lambda^g$ and $\mu \propto (E^\star)^{1-g}$, which yields formula (2.13). The exact $E^\star - \mu$ relation can be extracted from the results of the Bethe ansatz solution of the anisotropic Kondo model [24, 25] (see also [22]):

$$E^\star = (1-g)\,\frac{\sqrt{\pi}\,\Gamma\big(1+\frac{g}{2(1-g)}\big)}{\Gamma\big(\frac{1}{2}+\frac{g}{2(1-g)}\big)}\,\big(\,\Gamma(1-g)\,\mu\,\big)^{\frac{1}{1-g}}\ . \tag{2.17}$$

2.3 Fidelities $A_{21}(\mathcal{O}_\pm)$ and $A_{21}(\sigma_{0,3})$

Let us slightly generalize the setting from the introduction and consider the partition function $\mathcal{Z}_{21}\big(\mathcal{O}|y\big)$ of the half-infinite system with the insertion of a pair of hermitian conjugate boundary fields $\mathcal{O}$ and $\mathcal{O}^\dagger$ at the ends of the boundary segment where BC depends upon two energy scales $(E_2^\star, h_2)$ (see Fig. 3). In what follows we will make use of the notation

$$\bar{\mathcal{Z}}_{21}\big(\mathcal{O}|y\big) = \frac{\mathcal{Z}_{21}(\mathcal{O}|y)}{\mathcal{Z}_{11}}\ , \tag{2.18}$$

where $\mathcal{Z}_{11}$ is the partition function of the system without any boundary insertions and whose BC is defined by $(E_1^\star, h_1)$ homogeneously along the whole boundary. Of course, $\bar{\mathcal{Z}}_{21}(\mathcal{O}|y)$ is a complicated function and it is a challenging problem to compute it in a compact and manageable form even for the simplest boundary fields. However, since the physics of the model (2.11) is well understood now, one can make some general predictions regarding its behavior for small and large values of y. In what follows we will mostly discuss the simplest case with $\mathcal{O}$ given by the diagonal matrixes: σ_0, σ_3, or

$$\mathcal{O}_\pm = \frac{1}{2}\,(\sigma_0 \pm \sigma_3)\ . \tag{2.19}$$

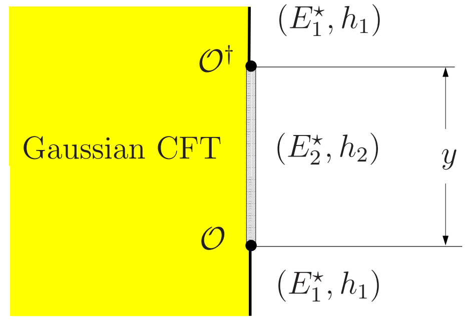

Figure 3: An universal part of $|\langle\, \Omega_2\,|\,\mathcal{O}(0)\,|\,\Omega_1\,\rangle|$ can be extracted from the partition function of the system with inhomogeneous boundary, and with the insertion of a pair of hermitian conjugate boundary fields $\mathcal{O}$ and $\mathcal{O}^\dagger$.

Since the model is asymptotically free at short distances, the effect of the boundary energy scales $(E_2^\star, h_2)$ from the interval $(0, y)$ becomes negligible wherein its size shrinks to zero, and therefore

$$\lim_{y\to 0^+} \bar{\mathcal{Z}}_{21}\big(\mathcal{O}|y\big) = \begin{cases} 1 & \text{for} \quad \mathcal{O} = \sigma_0,\, \sigma_3 \\ \frac{1}{2}\,\big(1 \pm m_1\big) & \text{for} \quad \mathcal{O} = \mathcal{O}_\pm \end{cases} . \tag{2.20}$$

Here we use

$$m = \frac{\langle\,\Omega\,|\,\sigma_3\,|\,\Omega\rangle}{\langle\,\Omega\,|\,\Omega\rangle} \tag{2.21}$$

and its subscript “1” shows that the expectation value is taken over the ground state $|\,\Omega_1\rangle$ corresponding to $(E_1^\star, h_1)$. Note that m coincides with the impurity magnetization in the context of the anisotropic Kondo model. It can be expressed in terms of the exact ground state energy E_0 of the Hamiltonian (2.11) as

$$m = -\bigg(\frac{\partial E_0}{\partial h}\bigg)_{E^\star} . \tag{2.22}$$

The spin degrees of freedom are freezing at large distances so that the large-y asymptotic of $\bar{\mathcal{Z}}_{21}\big(\mathcal{O}|y\big)$ has a form similar to (1.1) with the same exponent $d_{21} = d_{21}\big(\{E_i^\star, h_i\}\big)$ for any choice of the diagonal matrix $\mathcal{O} \in \big\{\sigma_0, \sigma_3, \mathcal{O}_\pm\big\}$:

$$\bar{\mathcal{Z}}_{21}\big(\mathcal{O}|y\big) = \big|A_{21}\big(\mathcal{O}\big)\big|^2\ \ y^{-2d_{21}}\ \mathrm{e}^{-y\Delta E_{21}}\ \big(1 + o(1)\big) \quad \text{as} \ \ y \to +\infty\ , \tag{2.23}$$

where $\Delta E_{21} = E_0(E_2^\star, h_2) - E_0(E_1^\star, h_1)$. Although (2.23) defines $A_{21}\big(\mathcal{O}_\pm\big)$ and $A_{21}\big(\sigma_{0,3}\big)$ in absolute value, their relative phases are dictated by the relations

$$A_{21}\big(\sigma_s\big) = A_{21}\big(\mathcal{O}_+\big) + (-1)^s\ A_{21}\big(\mathcal{O}_-\big) \qquad\qquad (s = 0, 3)\ . \tag{2.24}$$

In what follows, we will call $A_{21}\big(\mathcal{O}\big)$ as “fidelities” and treat them as scaling functions depending on the magnetic moments m_1, m_2 (2.22) as well as the pair of the Kondo temperatures $E_1^\star$, $E_2^\star$ (2.14). In fact, since they are dimensionful quantities, they depend non-trivially upon the dimensionless variables m_1, m_2 and $\alpha \equiv \log(E_2^\star/E_1^\star)$ only.

3 Mean field, perturbation theory and Toulouse limit

In this section we outline several common approaches for study of the fidelities in the spin-boson model.

3.1 Mean field approximation

We start with the model which is considerably simpler than the spin-boson model. The simplified Hamiltonian is obtained from $\boldsymbol{H}$ (2.11) through the substitution of the Pauli matrix σ_3 by a constant m. It splits into two non-interacting parts:

$$H_1 = H_{\rm free} - \frac{m}{2}\,\Pi_B \quad : \quad \mathcal{H} \mapsto \mathcal{H} \tag{3.1}$$

and a 2×2 matrix $\boldsymbol{H}_2 = -h\,\sigma_3 - J\,\sigma_1$. This can be thought of as a mean field approximation, with the value of $m = m(h)$ given by the relation $m = -\frac{\partial E_0}{\partial h}$, where $E_0 = -\sqrt{J^2+h^2}$ is the lowest eigenvalue of $\boldsymbol{H}_2$.

To construct the vacuum state for the mean field Hamiltonian one can use the interaction picture with the term $\propto m$ in (3.1) is treated as an interaction. The unitary operator $U(t,t_0) = \mathrm{e}^{\mathrm{i}H_{\rm free}(t-t_0)}\,\mathrm{e}^{\mathrm{i}H_1(t-t_0)}$ can be calculated explicitly in this case:

$$U(t,t_0) = \mathrm{e}^{\frac{\mathrm{i}m}{2}\Phi_B(t)}\ \mathrm{e}^{-\frac{\mathrm{i}m}{2}\Phi_B(t_0)}\ , \tag{3.2}$$

where $\Phi_B(t) = \mathrm{e}^{\mathrm{i}H_{\rm free}t}\,\Phi(0,0)\,\mathrm{e}^{-\mathrm{i}H_{\rm free}t}$. The vacuum for (3.1) is obtained through the Euclidean time evolution of the state $|\,\mathrm{vac}\,\rangle$ (2.4). Thus the vacuum state for the whole mean field Hamiltonian is given by

$$|\,\Omega\,\rangle = \mathrm{e}^{\frac{\mathrm{i}m}{2}\,\Phi_B(0)}\ \lim_{t_0\to\infty}\ \mathrm{e}^{-\frac{\mathrm{i}m}{2}\,\Phi_B(t_0)}\ |\,\mathrm{vac}\,\rangle \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{1+m}\\ \sqrt{1-m}\end{pmatrix}\ , \tag{3.3}$$

where the limit is taken along imaginary time direction $y_0 = \mathrm{i}\,t_0 \to -\infty$ as it is shown in Fig. 2. Taking into account the defining property (2.3) of the state $|\,\mathrm{vac}\,\rangle$, one obtains

$$|\,\Omega\,\rangle = T_+\ |\,\mathrm{vac}\,\rangle \otimes |\uparrow\rangle + T_-\ |\,\mathrm{vac}\,\rangle \otimes |\downarrow\rangle\ , \tag{3.4}$$

where

$$T_\pm = \sqrt{\frac{1\pm m}{2}}\ \mathrm{e}^{\frac{\mathrm{i}m}{2}\,\Phi_B(0)}\ , \tag{3.5}$$

and we use the common notation for σ_3-eigenvectors. Similarly one has

$$\langle\,\Omega\,| = \langle\,\mathrm{vac}\,|\,Q_+ \otimes \langle\uparrow| + \langle\,\mathrm{vac}\,|\,Q_- \otimes \langle\downarrow|$$

with

$$Q_\pm = \big(T_\pm\big)^\dagger = \sqrt{\frac{1\pm m}{2}}\ \mathrm{e}^{-\frac{\mathrm{i}m}{2}\,\Phi_B(0)}\ . \tag{3.6}$$

It is easy to see now that the mean field approximation yields the relation

$$\mathcal{Z}_{21}(\mathcal{O}_\pm|y) \propto \langle\, \mathrm{vac}\,|\, \mathrm{e}^{-\mathrm{i}\omega\Phi_B(y)}\, \mathrm{e}^{+\mathrm{i}\omega\Phi_B(0)}\,|\, \mathrm{vac}\,\rangle \quad \text{with} \quad \omega = \tfrac{1}{2}\,(m_1 - m_2) \tag{3.7}$$

and therefore to eq.(1.5) for the IR singularity exponent. As for the fidelities $A_{21}\big(\mathcal{O}_\pm\big)$, it is worth to note the relation

$$\langle\, p'\,|\, Q_\pm^{(2)}\, T_\pm^{(1)}\,|\, p\,\rangle = A_{21}\big(\mathcal{O}_\pm\big)\ \delta_{2p'-2p, m_1-m_2}\ . \tag{3.8}$$

Within the mean field approximation $A_{21}\big(\mathcal{O}_\pm\big) = \frac{1}{2}\sqrt{(1\pm m_2)(1\pm m_1)}$, which is found to be an adequate approximation as $g \to 0$. Contrary to the fidelities, the formula $d_{21} = \frac{g}{4}(m_2 - m_1)^2$ is expected to be exact as m_1 and m_2 are understood as vacuum expectation values of σ_3 in the spin-boson model. Evidences in its favor are presented in the next two subsections.

3.2 Renormalized perturbation theory

Similarly to the case $h_1 = h_2 = 0$ considered in ref. [23], the fidelities $A_{21}\big(\mathcal{O}_\pm\big)$ with $h_1, h_2 \neq 0$ can be calculated by means of the renormalized perturbation theory in coupling g for the Hamiltonian (1.4). The result of perturbative calculations turn out to be in agreement with the orthogonality exponent $d_{21} = \frac{g}{4}(m_2 - m_1)^2$. This ensure that $A_{21}\big(\mathcal{O}_\pm\big)$ can be written in the form

$$A_{21}\big(\mathcal{O}_\pm\big) = \big(E_2^\star\big)^{\frac{g}{4}(m_2\mp 1)(m_1-m_2)}\ \big(E_1^\star\big)^{\frac{g}{4}(m_1\mp 1)(m_2-m_1)}\ \ \mathcal{A}_\pm(m_2, m_1|\,\alpha)\ , \tag{3.9}$$

where

$$\alpha \equiv \log\big(E_2^\star/E_1^\star\big)\ , \tag{3.10}$$

and the prefactor has a dimension of $[\mathit{energy}\,]^{-d_{21}}$, so that $\mathcal{A}_\pm$ are dimensionless amplitudes. Notice that in the case $\mu_2 = \mu_1$, $h_2 = h_1$, the orthogonality exponent vanishes and $\mathcal{A}_\pm$ should satisfy an exact relation

$$\mathcal{A}_\pm(m, m\,|\,0\,) = \frac{1}{2}\,(1 \pm m)\ . \tag{3.11}$$

It can be shown with somewhat cumbersome but straightforward effort that

$$\begin{aligned}\mathcal{A}_+(m_2, m_1\,|\,\alpha) = {}& \frac{1}{2}\ \sqrt{(1+m_2)(1+m_1)}\ \ \big(2\mathrm{e}^{\gamma_E}\big)^{-\frac{g}{4}(m_1-m_2)^2} \\ &\times\ \big(1-m_2^2\big)^{\frac{g}{8}(1-m_2)(m_1-m_2)}\ \big(1-m_1^2\big)^{\frac{g}{8}(1-m_1)(m_2-m_1)} \\ &\times\ \left(1 + \frac{g}{4}\,\Big((1-m_1)^2 + (1-m_2)^2 - (1-m_1)(1-m_2)\,\delta\,\coth\frac{\delta}{2}\Big) + O(g^2)\right)\end{aligned} \tag{3.12}$$

and

$$\mathcal{A}_-(m_2, m_1\,|\,\alpha) = \mathcal{A}_+(-m_2, -m_1\,|\,\alpha)\ . \tag{3.13}$$

Here γ_E stands for the Euler constant, and we use $\delta \equiv \alpha + \frac{1}{2}\log\big(\frac{1-m_1^2}{1-m_2^2}\big)$. The quoted result shows that the perturbative amplitudes $\mathcal{A}_\pm$ are multivalued functions of the complex variables

$(m_2, m_1|\,\alpha)$. However their overall phases can be chosen in such a way that they are real analytic within the domain

$$\mathbb{D} \equiv \big\{ \, (m_1, m_2 \,|\, \alpha) \ : \ m_{1,2} \in (-1, 1) \, ; \ \alpha \in \mathbb{R} \, \big\} \ . \tag{3.14}$$

It is also easy to see that $\mathcal{A}_\pm$ satisfy the condition

$$\mathcal{A}_\pm(m_2, m_1 \,|\, \alpha) = \mathcal{A}_\pm(m_1, m_2 \,|\, -\alpha) \ . \tag{3.15}$$

Besides, the use of the perturbation theory allows one to determine the relation between m and dimensionless ratio $h/E^\star$ in a form of a power series in g. In particular, to the first order, one has

$$\frac{h}{E^\star} = \frac{m}{\sqrt{1-m^2}} \, \Big(1 - \frac{g}{2} \, \log(1-m^2) + O(g^2) \Big) \ . \tag{3.16}$$

With this formula the perturbative amplitudes can be expressed in terms of $h_i/E^\star_i$ $(i = 1, 2)$. Notice that, from the Bethe ansatz solution of the anisotropic Kondo model, it is known that [24, 25]

$$m = \frac{\mathrm{i}}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \lambda^{\frac{\mathrm{i}\omega}{1-g}} \ \frac{M(\omega)}{\omega + \mathrm{i}0} \ , \qquad M(\omega) = \frac{\Gamma(1 - \frac{\mathrm{i}\omega}{2(1-g)})\Gamma(\frac{1}{2} + \frac{\mathrm{i}\omega}{2})}{\sqrt{\pi} \ \Gamma(1 - \frac{\mathrm{i}\omega g}{2(1-g)})} \, \Big(\Gamma(1-g)\Big)^{\frac{\mathrm{i}\omega}{1-g}} \ , \tag{3.17}$$

where

$$\lambda = \frac{1}{\Gamma(1-g)} \left[\frac{\Gamma\big(\frac{1}{2} + \frac{g}{2(1-g)}\big)}{\sqrt{\pi}(1-g)\Gamma\big(1 + \frac{g}{2(1-g)}\big)} \right]^{1-g} \left(\frac{E^\star}{h}\right)^{1-g} . \tag{3.18}$$

This remarkable exact result implies the following general structure of the perturbative expansion (3.16):

$$\frac{h}{E^\star} = m \, (1-m^2)^{-\frac{1}{2-2g}} \left(1 + \sum_{n=2}^{\infty} g^n \sum_{l=1}^{n-1} c_{nl} \, m^{2l} \right) , \tag{3.19}$$

where c_{nl} are some numerical coefficients.

3.3 Toulouse limit

In the case $g = \frac{1}{2}$, which is sometimes referred as to "Toulouse limit", the Hamiltonian $\boldsymbol{H}_\phi$ (2.15) can be fermionized in terms of the chiral complex fermion field

$$\boldsymbol{H}_{\text{Toul}} = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}x \ \psi^\dagger \partial_x \psi - \mu \left(\hat{d}^\dagger \psi(0) + \psi^\dagger(0) \hat{d} \right) + h \, (\hat{d}^\dagger \hat{d} - \hat{d} \hat{d}^\dagger) \ , \tag{3.20}$$

where $\{\psi^\dagger(x), \psi(x')\} = \delta(x - x')$, $\{\hat{d}^\dagger, \hat{d}\} = 1$, *etc.* Equivalently, the model can be understood as a boundary flow for the Dirac fermion, massless in the bulk. In order to construct an Euclidean action governing this boundary flow, we define $\bar{\psi}(x, y) \equiv \psi(-x, y)$, so that the

Hamiltonian (3.20) for $\mu = h = 0$ takes the form $\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{0}\mathrm{d}x\ \big(\psi^\dagger\partial_x\psi - \bar\psi^\dagger\partial_x\bar\psi\big)$. The fields ψ and $\bar\psi$ are interpreted now as components of the massless Dirac fermion, both defined on the half-infinite lane $x \le 0$ and satisfying the bulk equations of motion $\partial_{\bar z}\psi = \partial_z\bar\psi = 0$ with $z = x+\mathrm{i}y$, $\bar z = x-\mathrm{i}y$. The complex fermions can be substituted by two Majorana-Weyl fermions: $\psi = \frac{\psi_1+\mathrm{i}\psi_2}{\sqrt 2}$, $\bar\psi = \frac{\bar\psi_1+\mathrm{i}\bar\psi_2}{\sqrt 2}$. Each of the real fermions $(\psi_j, \bar\psi_j)$ satisfies the free BC, $\big(\psi_j-\bar\psi_j\big)|_{x=0} = 0$ and, as it was explained in ref. [26], should be described by means of the action $\mathcal{A}_{\rm MW}[\psi_j,\bar\psi_j,a_j]$ which involves additional boundary fermionic degree of freedom $a_j = a_j(y)$:

$$\mathcal{A}_{\rm MW} = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}y\int_{-\infty}^{0}\mathrm{d}x\ \big(\psi_j\partial_{\bar z}\psi_j - \bar\psi_j\partial_z\bar\psi_j\big) + \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}y\Big(\frac{1}{2\pi\mathrm{i}}(\psi_j\bar\psi_j)|_{x=0} + a_j\partial_y a_j\Big). \quad (3.21)$$

An Euclidean action, corresponding to the Hamiltonian (3.20) with non-vanishing couplings μ and h, is given by [26][1]

$$\mathcal{A}_{\rm Toul} = \sum_{j=1}^{2}\mathcal{A}_{\rm MW}[\psi_j,\bar\psi_j,a_j] + \int_{-\infty}^{\infty}\mathrm{d}y\ \Big(\mu\,\mathfrak{S}_B(y) - h\ \mathfrak{M}_B(y)\Big)\ , \quad (3.22)$$

where

$$\mathfrak{S}_B = \frac{\mathrm{i}}{2}\ \sum_{j=1}^{2} a_j\ \big(\psi_j + \bar\psi_j\big)|_{x=0}\ , \qquad \mathfrak{M}_B = 2\mathrm{i}\ a_1 a_2\ . \quad (3.23)$$

Clearly, the Grassmannian boundary fields $d = \frac{a_1+\mathrm{i}a_2}{\sqrt 2}$, $d^* = \frac{a_1-\mathrm{i}a_2}{\sqrt 2}$ in the path integral formalism correspond to the nilpotent operators $\hat d$ and $\hat d^\dagger$ in the Hamiltonian picture.

The general solution of the bulk equations, $\partial_{\bar z}\psi = \partial_z\bar\psi = 0$, are given by the Fourier integrals

$$\psi = \int_{-\infty}^{\infty}\mathrm{d}k\ c_k\ \mathrm{e}^{\mathrm{i}kz}\ , \qquad \bar\psi = \int_{-\infty}^{\infty}\mathrm{d}k\ \bar c_k\ \mathrm{e}^{-\mathrm{i}k\bar z}\ . \quad (3.24)$$

In their turn, the boundary equations of motion corresponding to the action (3.22) allows one to express the Fourier modes $\bar c_k$ and the Heisenberg operator $\hat d(y)$ in terms of c_k:

$$\bar c_k = \mathrm{e}^{2\mathrm{i}\delta(k)}\ c_k \qquad \text{with} \qquad \mathrm{e}^{2\mathrm{i}\delta(k)} = \frac{k - \mathrm{i}\pi\mu^2 - 2h}{k + \mathrm{i}\pi\mu^2 - 2h} \quad (3.25)$$

and

$$\hat d(y) = \mathrm{i}\ \mu\ \int_{-\infty}^{\infty}\mathrm{d}k\ \frac{c_k\ \mathrm{e}^{-ky}}{k + \mathrm{i}\pi\mu^2 - 2h}\ . \quad (3.26)$$

With this, it is straightforward to compute the Euclidean propagator for the complex boundary fermions:

$$\langle\, d^*(y)\, d(0)\,\rangle = D(y|-h)\ \Theta(y) - \Theta(-y)\ D(y|h)\ , \quad (3.27)$$

[1] Although in ref. [26] the case of a single Majorana-Weyl fermion was only discussed, Eq.(3.22) is an apparent consequence of the Chatterjee-Zamolodchikov result.

where $\Theta(y) = \frac{1}{2}\big(1+\text{sign}(y)\big)$ and

$$D(y|h) = \mu^2 \int_0^\infty \mathrm{d}k \ \frac{\mathrm{e}^{-k|y|}}{(k-2h)^2+\pi^2\mu^4} \to \begin{cases} \frac{1+m}{2} & \text{as} \quad y \to 0 \\ \frac{\cos^2(\frac{\pi m}{2})}{4E^\star y} & \text{as} \quad y \to \infty \end{cases} . \tag{3.28}$$

In the last formula we use the notations $E^\star = (\frac{\pi\mu}{2})^2$ and

$$m = \frac{2}{\pi} \ \arctan\Big(\frac{\pi h}{2E^\star}\Big) \ , \tag{3.29}$$

which are consistent with general relations (2.17) and (3.17) taken at $g = \frac{1}{2}$.

In order to calculate the orthogonality exponent, one needs to introduce an explicit IR regularization. Let us restrict values of the space coordinate x to the segment $[-L, 0]$ and choose the free BC at $x = -L$: $\big(\psi - \bar{\psi}\big)|_{x=-L} = \big(\psi^\dagger - \bar{\psi}^\dagger\big)|_{x=-L}$, or equivalently, $\tilde{c}_k \ \mathrm{e}^{\mathrm{i}Lk} = c_k \ \mathrm{e}^{-\mathrm{i}Lk}$. Taking this together with eq.(3.25), one obtains the quantization condition $2Lk_n + 2\,\delta(k_n) = 2\pi n$ ($n \in \mathbb{Z}$), so that Fourier integral expansions (3.24) should be substituted by discrete sums $\psi = \sum_{n=-\infty}^{\infty} \mathsf{c}_n \ \mathrm{e}^{\mathrm{i}k_n(x-t)}$, and similar for $\bar{\psi}$. Note that the real-time evolution of the fermion modes c_n is produced by the Hamiltonian $H = \sum_{n=-\infty}^{+\infty} k_n \, \mathsf{c}_n^\dagger \mathsf{c}_n$ through the canonical commutation relations $\{\mathsf{c}_n^\dagger, \mathsf{c}_{n'}\} = \delta_{n,n'}$, $\mathsf{c}_n^2 = (\mathsf{c}_n^\dagger)^2 = 0$. A ground state of the system of fermionic oscillators is defined by the requirement that all energy levels bellow the Fermi level $k_F = 0$ are occupied. In this situation, according to Anderson [2], the orthogonality exponent for the ground state overlap $\langle\, \Omega_2 \,|\, \Omega_1 \,\rangle$ is determined by the the difference of the phase shifts at the Fermi level:

$$d_{21}\big|_{g=\frac{1}{2}} = \frac{1}{2\pi^2} \ \big(\delta_2(0) - \delta_1(0)\big)^2 \ . \tag{3.30}$$

In its turn, the phase shift $\delta(k)$ (3.25) at $k = 0$ can be written in terms of m (3.29) as $\delta(0) = \frac{\pi}{2}\ (1-m)$, and therefore $d_{21} = \frac{1}{8}\ (m_2 - m_1)^2$. This coincides with (1.5) specialized at $g = \frac{1}{2}$.

It is useful to note that the orthogonality exponent can be written in the form

$$d_{21} = \frac{1}{2}\ (q_1 - q_2)^2 \ , \tag{3.31}$$

where $q_i = \langle \Omega_i \,|\, \hat{q} \,|\, \Omega_i \,\rangle$ are vacuum expectation values of the operator $\hat{q} = \sum_{n=-\infty}^{+\infty} c_n^\dagger c_n$. Numerical results presented in ref. [18] suggest that eq.(3.31) remains valid for $g \neq \frac{1}{2}$. For their calculations, the authors used the resonant level model [12] whose Hamiltonian is obtained by adding a four-fermion interaction term to $\boldsymbol{H}_{\text{Toul}}$ (3.20),

$$\boldsymbol{H}_{\text{RLM}} = \boldsymbol{H}_{\text{Toul}} + u\,(\hat{d}^\dagger \hat{d} - \hat{d}^\dagger \hat{d}) \ \ :\psi^\dagger\psi(0): \ . \tag{3.32}$$

As it is well known (see, e.g., [27, 28]), this is a fermionic version of the Hamiltonian $\boldsymbol{H}_\phi$ for general values of $g \in [0,1]$. The difference $q_1 - q_2$ in (3.31) was referred in ref. [18] as to the "displaced charge", which is, in a sense, the difference between vacuum expectation values of the charges associated with the global $U(1)$ symmetry of the resonant level model. We may now note that formulae (1.5) and (3.31) for the orthogonality exponent coincides provided

$$q_1 - q_2 = \sqrt{\frac{g}{2}}\ (m_2 - m_1) \ . \tag{3.33}$$

The last relation indeed holds true for the resonant level model (see Chapter 28IV.2 in ref. [27] for details).

Returning to the spin-boson model at the Toulouse limit, we acknowledge the relations between the normalized partition functions $\bar{\mathcal{Z}}_{11}(\sigma_{3,1}|y) \equiv \bar{\mathcal{Z}}_{21}(\sigma_{3,1}|y)|_{E_2^\star=E_1^\star, h_2=h_1}$ and two-point functions of the boundary fields (3.23):

$$\begin{aligned} \bar{\mathcal{Z}}_{11}\big(\sigma_3|y\big)\Big|_{g=\frac{1}{2}} &= \langle\, \mathfrak{M}_B(y)\, \mathfrak{M}_B(0)\,\rangle \\ \bar{\mathcal{Z}}_{11}\big(\sigma_1|y\big)\Big|_{g=\frac{1}{2}} &= \langle\, \mathfrak{S}_B(y)\, \mathfrak{S}_B(0)\,\rangle\ . \end{aligned} \tag{3.34}$$

The boundary equations of motion corresponding to the action (3.22), allows one to represent the Heisenberg operators $\hat{\mathfrak{M}}_B(y)$ and $\hat{\mathfrak{S}}_B(y)$ in the form of normal-ordered combinations of $\hat{d}(y)$ and its Hermitian conjugates:

$$\begin{aligned} \hat{\mathfrak{M}}_B &= 2 : \hat{d}\hat{d}^\dagger : + m \\ \hat{\mathfrak{S}}_B &= \mu^{-1}\ : \big(\, (\partial_y \hat{d})\, \hat{d}^\dagger + (\partial_y \hat{d}^\dagger)\hat{d} - 4h\, \hat{d}^\dagger \hat{d}\big) : + \langle\, \mathfrak{S}_B\,\rangle\ , \end{aligned} \tag{3.35}$$

where the vacuum expectation value,

$$\langle\, \mathfrak{S}_B\,\rangle = \tfrac{2}{\pi}\, \sqrt{E^\star}\, \Big[2\, \log\big(\tfrac{E^\star}{\Lambda}\big) + \log\Big(1 + \big(\tfrac{h}{E^\star}\big)^2\Big)\Big]\ , \tag{3.36}$$

contains the non-universal term which depends on the UV cutoff scale Λ. Using the Wick theorem, the two-point functions (3.34) can be written in terms of $D_\pm \equiv D(y|\pm h)$ (3.28),

$$\begin{aligned} &\langle\, \mathfrak{M}_B(y)\, \mathfrak{M}_B(0)\,\rangle = m^2 + 4 D_- D_+ \\ &\langle\, \mathfrak{S}_B(y)\, \mathfrak{S}_B(0)\,\rangle_{\rm conn} = \frac{D_- D_+}{\mu^2}\, \bigg(\frac{\ddot{D}_-}{D_-} + \frac{\ddot{D}_+}{D_+} - 2\, \frac{\dot{D}_- \dot{D}_+}{D_- D_+} + 8h\Big(\frac{\dot{D}_+}{D_+} - \frac{\dot{D}_-}{D_-}\Big) + 16 h^2\bigg). \end{aligned} \tag{3.37}$$

Here the dot stands for the derivative w.r.t. the Euclidean time y and the abbreviation "conn" means the connected correlation function: $\langle\, \mathfrak{S}_B(y)\, \mathfrak{S}_B(0)\,\rangle_{\rm conn} \equiv \langle\, \mathfrak{S}_B(y)\, \mathfrak{S}_B(0)\,\rangle - \langle\, \mathfrak{S}_B\,\rangle^2$. Similarly, one has

$$\langle\, \mathfrak{S}_B(y)\, \mathfrak{M}_B(0)\,\rangle = \frac{2 D_- D_+}{\mu}\, \bigg(\frac{\dot{D}_+}{D_+} - \frac{\dot{D}_-}{D_-} - 4h\bigg)\ . \tag{3.38}$$

Since the two-point functions (3.37), (3.38) are available in closed forms, the partition function $\bar{\mathcal{Z}}_{21}\big(\sigma_0|y\big)$ can be calculated perturbatively in powers of $\delta m \equiv m_2 - m_1 \ll 1$ and $\delta\alpha \equiv (E_2^\star - E_1^\star)/E_1^\star \ll 1$. In the case $\delta m = m = 0$, details of the calculations may be found in Appendix in ref. [23]. Similar calculations for non-zero m and δm yield the first non-vanishing terms of the Taylor expansion of the fidelity $A_{21}(\sigma_0)$ in the Toulouse limit:

$$A_{21}(\sigma_0)|_{g=\frac{1}{2}} = \bigg(\frac{\pi\cos(\frac{\pi m}{2})}{4 e^{\gamma_E} E^\star}\bigg)^{d_{21}}\ \bigg(1 - \chi_{\alpha\alpha}\ \frac{(\delta\alpha)^2}{2} - \chi_{mm}\ \frac{(\delta m)^2}{2} + O(\delta^3)\bigg)\ . \tag{3.39}$$

Here $m = m_1$, $E^\star = E_1$, $d_{21} = (\delta m)^2/16$ and

$$\begin{aligned} \chi_{\alpha\alpha} &= \frac{1}{4\pi^2}\, \bigg(1 + \frac{\pi}{2}(m+1)\ \tan\Big(\frac{\pi m}{2}\Big)\bigg)\bigg(1 + \frac{\pi}{2}(m-1)\ \tan\Big(\frac{\pi m}{2}\Big)\bigg) \\ \chi_{mm} &= \frac{\pi^2\ \chi_{\alpha\alpha}}{\sin^2(\pi m)} - \frac{\cos(\pi m)}{16\sin^2(\frac{\pi m}{2})}\, \bigg(1 + \pi m\, \tan\Big(\frac{\pi m}{2}\Big)\bigg)\ . \end{aligned} \tag{3.40}$$

4 Jost operators

In this section we introduce the notion of quantum Jost operators and discuss their properties.

4.1 Classical Jost functions

Let us first consider the limit $g \to 0$ where the boundary field $\Pi_B(t) = \dot{\Phi}_B(t)$ is treated as a classical field. In this approximation the Hamiltonian (2.11) describes the quantum spin in the presence of a classical background field. The corresponding non-stationary Schrödinger equation has the form

$$\mathrm{i}\,\frac{\partial \Psi}{\partial t} = -\big(\, J\,\sigma_1 + \dot{\phi}_c(t)\,\sigma_3\,\big)\ \Psi\ , \tag{4.1}$$

where we use $\phi_c(t) = \frac{1}{2}\,\Phi_B(t) + ht$, satisfying the asymptotic condition

$$\phi_c(t) \to ht + o(1) \qquad \text{as} \quad t \to -\infty \tag{4.2}$$

with $h > 0$. Let $\boldsymbol{U}(t, t_0)$ be a time evolution matrix for (4.1). It can be written as

$$\boldsymbol{U}(t, t_0) = \mathrm{e}^{\mathrm{i}\sigma_3\phi_c(t)}\boldsymbol{S}(t, t_0)\ \mathrm{e}^{-\mathrm{i}\sigma_3\phi_c(t_0)}\ , \tag{4.3}$$

where $\boldsymbol{S}$ stands for a time-ordered matrix exponential

$$\boldsymbol{S}(t, t_0) = \mathcal{T}_t \exp\left(\mathrm{i}\int_{t_0}^{t} \mathrm{d}t\, J\,\big(\,\mathrm{e}^{2\mathrm{i}\phi_c(t)}\,\sigma_- + \mathrm{e}^{-2\mathrm{i}\phi_c(t)}\,\sigma_+\,\big)\right)\ . \tag{4.4}$$

Consider the time evolution of the spin-up state $|\uparrow\rangle$ starting from the initial moment t_0. The following limiting behavior for $t_0 \to -\infty$, can be easily established:

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\phi_c(t)\sigma_3}\boldsymbol{S}(t, t_0)\,|\uparrow\rangle \ \ &\to\ \ \left(\frac{\mathtt{k}+h}{2\mathtt{k}}\ \Psi_{+\mathtt{k}}(t)\,|\uparrow\rangle + \frac{J}{2\mathtt{k}}\ \tilde{\Psi}_{+\mathtt{k}}(t)\,|\downarrow\rangle\right)\mathrm{e}^{-\mathrm{i}(\mathtt{k}-h)t_0} \\ &+\ \ \left(\frac{\mathtt{k}-h}{2\mathtt{k}}\ \Psi_{-\mathtt{k}}(t)\,|\uparrow\rangle - \frac{J}{2\mathtt{k}}\ \tilde{\Psi}_{-\mathtt{k}}(t)\,|\downarrow\rangle\right)\mathrm{e}^{\mathrm{i}(\mathtt{k}+h)t_0}\ . \end{aligned} \tag{4.5}$$

Here $\Psi_{\pm\mathtt{k}}(t)$, $\tilde{\Psi}_{\pm\mathtt{k}}(t)$ are the Jost solutions of the Sturm-Liouville equations

$$\begin{aligned} \big(-\partial_t^2 + U(t)\,\big)\ \Psi_{\pm\mathtt{k}} &= J^2\,\Psi_{\pm\mathtt{k}}\ , \qquad U = -\dot{\phi}_c^2 + \mathrm{i}\,\ddot{\phi}_c \\ \big(-\partial_t^2 + \tilde{U}(t)\,\big)\ \tilde{\Psi}_{\pm\mathtt{k}} &= J^2\,\tilde{\Psi}_{\pm\mathtt{k}}\ , \qquad \tilde{U} = -\dot{\phi}_c^2 - \mathrm{i}\,\ddot{\phi}_c\ , \end{aligned} \tag{4.6}$$

satisfying the asymptotic conditions at $t \to -\infty$:

$$\Psi_{\pm\mathtt{k}}(t) \to \mathrm{e}^{\pm\mathrm{i}\mathtt{k}t}\ , \qquad \tilde{\Psi}_{\pm\mathtt{k}}(t) \to \mathrm{e}^{\pm\mathrm{i}\mathtt{k}t}\ , \tag{4.7}$$

where $\mathtt{k} = \sqrt{J^2 + h^2} > 0$. Because of the presence of oscillating phase factors, the r.h.s. of (4.5) does not possess a finite limit as t_0 tends to $-\infty$. However, if we assume that the coupling J is switched off adiabatically,

$$\lim_{t_0 \to -\infty} J(t_0) = 0\ , \tag{4.8}$$

then $\lim_{t_0\to-\infty} \mathtt{k} = h > 0$ and the first term in (4.5) has a finite limit. The second term will still oscillate $\propto \mathrm{e}^{2\mathrm{i}ht_0}$ as $t_0 \to -\infty$. With the aim to suppress these oscillations, let us fix the value of t, say $t = 0$, and assume that the asymptotic behavior (4.5) holds true in the infinitesimal wedge $0 < \arg(-t_0) < \epsilon \to +0$ of complex plane t_0. Then, taking the limit along any ray inside the wedge, one obtains

$$\lim_{\substack{|t_0|\to+\infty \\ \arg(-t_0)=+0}} \Big(\mathrm{e}^{\mathrm{i}\phi_c(0)\sigma_3}\, \boldsymbol{S}(0,t_0)\,|\uparrow\rangle\Big) = \sqrt{\frac{1+m}{2}}\,\Big(T_+^{(c)}\,|\uparrow\rangle + T_-^{(c)}\,|\downarrow\rangle\Big)\ , \tag{4.9}$$

where we use $m = \frac{h}{\sqrt{J^2+h^2}}$. The t-independent connection coefficients

$$T_+^{(c)} = \sqrt{\frac{1+m}{2}}\ \ \Psi_{+\mathtt{k}}(0)\ , \qquad T_-^{(c)} = \sqrt{\frac{1-m}{2}}\ \ \tilde{\Psi}_{+\mathtt{k}}(0) \tag{4.10}$$

are sometimes referred as to Jost functions. It is easy to see that eqs.(4.6), (4.7) imply that

$$\tilde{\Psi}_{\pm\mathtt{k}}(t) = \frac{\mathrm{i}}{h\mp\mathtt{k}}\,\big(\partial_t - \mathrm{i}\dot{\phi}_c\big)\Psi_{\pm\mathtt{k}}(t)\ , \quad \Psi_{\pm\mathtt{k}}(t) = -\frac{\mathrm{i}}{h\pm\mathtt{k}}\,\big(\partial_t + \mathrm{i}\dot{\phi}_c\big)\tilde{\Psi}_{\pm\mathtt{k}}(t)$$

and also $\tilde{\Psi}_{\mp\mathtt{k}}(t) = \Psi^*_{\pm\mathtt{k}}(t)$ for real J, h and t. Using this and also taking into account that the Wronskian of $\Psi_{-\mathtt{k}}(t)$ and $\Psi_{+\mathtt{k}}(t)$ equals $2\mathrm{i}\mathtt{k}$, one finds the bilinear relation

$$\big(T_+^{(c)}\big)^* T_+^{(c)} + \big(T_-^{(c)}\big)^* T_-^{(c)} = 1\ . \tag{4.11}$$

Comparing (4.10) with eqs.(3.5), (3.6) we note that within the mean field approximation, the classical connection coefficients $\Psi_{+\mathtt{k}}(0)$ and $\tilde{\Psi}_{+\mathtt{k}}(0)$ are substituted by the exponential operators $\mathrm{e}^{\frac{\mathrm{i}m}{2}\Phi_B}(0)$ and $\mathrm{e}^{-\frac{\mathrm{i}m}{2}\Phi_B}(0)$, respectively.

4.2 Anticipated properties of Jost operators

Motivated by the above consideration, we start from the Hamiltonian (2.15) which describes an interaction of a local spin impurity with a chiral bose field on the whole line. Consider the interaction picture, treating the term $\propto \mu$ as a perturbation. Let us perform the Wick rotation from the very beginning, so that $\phi(x,y) = \phi(x+\mathrm{i}y)$ for the chiral bose field in the interaction picture and introduce the y-ordered exponent

$$\boldsymbol{S}(y_2,y_1\,|\,\mu_L,h) = \mathcal{T}_y \exp\left(\int_{y_1}^{y_2} \mathrm{d}y\ \mu_L(y)\ \big(\mathrm{e}^{2\mathrm{i}\phi(0,y)}\,\mathrm{e}^{2hy}\sigma_- + \mathrm{e}^{-2\mathrm{i}\phi(0,y)}\,\mathrm{e}^{-2hy}\ \sigma_+\big)\right)\ . \tag{4.12}$$

Here the renormalized coupling μ is substituted by y-dependent function, $\mu_L(y) > 0$ such that $\mu = \mu_L(0)$, and which is switching off adiabatically within an Euclidean time interval $|y| < L$. For technical reason, it is also convenient to choose the smoothing function to be an even function of y. The explicit IR regularization allows one to define

$$\begin{aligned} \boldsymbol{U}^{(-)}(\mu_L,h) &= \mathrm{e}^{\mathrm{i}\phi(0,0)\sigma_3}\,\boldsymbol{S}(0,-\infty\,|\,\mu_L,h) \\ \boldsymbol{U}^{(+)}(\mu_L,h) &= \boldsymbol{S}(+\infty,0\,|\,\mu_L,h)\,\mathrm{e}^{-\mathrm{i}\phi(0,0)\sigma_3}\ , \end{aligned} \tag{4.13}$$

which are operators acting on the impurity spin, whose matrix elements are themselves operators acting on the free bosonic degrees of freedom, i.e. in the Hilbert space $\mathcal{H}$ (2.5). In the absence of interaction and for $h>0$, the vacuum state is given by the product $|\,\mathrm{vac}\,\rangle\otimes|\uparrow\rangle$. Naively, the vacuum in the interacting theory occurs through the large-L limit:

$$\begin{aligned}
\Big[\mathrm{e}^{\Delta f(L)}\ \ \boldsymbol{U}^{(-)}(\mu_L,h)\,|\,\mathrm{vac}\,\rangle\otimes|\uparrow\rangle\Big]_{L\to+\infty} &\ \to\ |\,\Omega\,\rangle \\
\Big[\mathrm{e}^{\Delta f(L)}\ \ \langle\,\mathrm{vac}\,|\otimes\langle\uparrow|\,\boldsymbol{U}^{(+)}(\mu_L,h)\Big]_{L\to+\infty} &\ \to\ \langle\,\Omega\,|\ ,
\end{aligned}\tag{4.14}$$

where $\Delta f(L)=\int_{-\infty}^{0}\mathrm{d}y\,\big(E_0\big(\mu_L(y),h\big)+h\big)$. But, because the IR problem, these asymptotic relations cannot be literally true. We will try to overcome this obstacle using a heuristic picture which is based on the notion of quantum Jost operators. Namely, we postulate that the exact vacuum state is given by relations similar to the one obtained in Sec. 3.1 within the mean field approximation, i.e.,

$$\begin{aligned}
|\,\Omega\,\rangle &= T_+(\mu,h)\,|\,\mathrm{vac}\,\rangle\otimes|\uparrow\rangle+T_-(\mu,h)\,|\,\mathrm{vac}\,\rangle\otimes|\downarrow\rangle \\
\langle\,\Omega\,| &= \langle\,\mathrm{vac}\,|\,Q_+(\mu,h)\otimes\langle\uparrow|+\langle\,\mathrm{vac}\,|\,Q_-(\mu,h)\otimes\langle\downarrow|\ ,
\end{aligned}\tag{4.15}$$

and T_ε and Q_ε, with $\varepsilon=\pm$ and $\mu,h>0$, are operators acting as

$$\begin{aligned}
T_\varepsilon(\mu,h)\ &:\ \ \mathcal{H}\mapsto\mathcal{H}\ \ \&\ \ \mathcal{F}_p\mapsto\mathcal{F}_{p+\frac{m}{2}} \\
Q_\varepsilon(\mu,h)\ &:\ \ \mathcal{H}\mapsto\mathcal{H}\ \ \&\ \ \mathcal{F}_p\mapsto\mathcal{F}_{p-\frac{m}{2}}\ ,
\end{aligned}\tag{4.16}$$

such that p-vacuum expectation values involving bilinear combinations of T and Q are expressed in terms of the dimensionless amplitudes $\mathcal{A}_\pm$ (3.9):

$$\langle\,p'\,|\,Q_\pm(\mu_2,h_2)\,T_\pm(\mu_1,h_1)\,|\,p\,\rangle=\mathcal{A}_\pm\big(m_2,\,m_1\,|\,\alpha\big)\ \delta_{2p'-2p,m_1-m_2}\ .\tag{4.17}$$

Notice that, since T and Q act invariantly in $\mathcal{H}$, the p-dependence appears here through the Kronecker delta only. In what follows T- and Q-operators are referred as to Jost operators. In order to predict their properties, we shall invoke the intuition which is based on the classical limit, the results of perturbative calculations and global symmetries of the model.

- ***T − invariance.*** The time reversal transformation acts as $\mathrm{e}^{\pm\mathrm{i}\phi(0,y)}\mapsto\mathrm{e}^{\mp\mathrm{i}\phi(0,-y)}$ and $\sigma_\pm\mapsto\sigma_\mp,\ \sigma_3\mapsto-\sigma_3$. The antiunitary operator defined by (2.7), satisfies the relation

  $$\mathbb{T}\big(\mathrm{e}^{\mathrm{i}a_1\phi_1(0,y_1)}\,\mathrm{e}^{\mathrm{i}a_2\phi(0,y_2)}\cdots\mathrm{e}^{\mathrm{i}a_n\phi_1(0,y_n)}\big)\mathbb{T}=\mathrm{e}^{\mathrm{i}a_n\phi(0,-y_n)}\ \cdots\ \mathrm{e}^{\mathrm{i}a_2\phi(0,-y_2)}\,\mathrm{e}^{-\mathrm{i}a_1\phi_1(0,-y_1)}$$

  for $y_1>y_2>\cdots>y_n$. Ignoring the problem with the IR divergency, one can expand the y-ordered exponent (4.12) and find

  $$\mathbb{T}\,T_\varepsilon\big(\mu,h\big)\,\mathbb{T}=Q_\varepsilon\big(\mu^*,h\big)\ ,\qquad \mathbb{T}\,Q_\varepsilon\big(\mu,h\big)\,\mathbb{T}=T_\varepsilon\big(\mu^*,h\big)\ .\tag{4.18}$$

  Notice that the T-invariance, when it is applied to (4.17), leads to the relation (3.15).

- ***Hermiticity.*** The Jost operators satisfy a hermitian conjugation condition

  $$Q_\varepsilon(\mu,h)=\big(T_\varepsilon(\mu,h)\big)^\dagger\qquad(\mu,\ h>0)\ ,\tag{4.19}$$

  which follows from the relations $\big(\mathrm{e}^{\mathrm{i}\phi(0,0)\sigma_3}\big)^\dagger=\mathrm{e}^{-\mathrm{i}\phi(0,0)\sigma_3}$, $\big(\mathrm{e}^{2\mathrm{i}\phi(0,y)}\,\sigma_-\big)^\dagger=\mathrm{e}^{-2\mathrm{i}\phi(0,-y)}\,\sigma_+$. Together with (3.15), the conjugation implies the reality condition $\big(\mathcal{A}_\pm(m_2,m_1|\alpha)\big)^*=\mathcal{A}_\pm(m_2,m_1|\alpha)$ for real positive μ_i, h_i $(i=1,2)$.

- ***C* − *invariance.*** The formal C-transformation acts as $\mathrm{e}^{2\mathrm{i}a\phi(0,y)} \mapsto \mathrm{e}^{-2\mathrm{i}a\phi(0,y)}$, $\sigma_\pm \mapsto \sigma_\mp$, $\sigma_3 \mapsto -\sigma_3$. It is well known that there is no spontaneous magnetization in the anisotropic Kondo model. Because of this we expect that there exists an operator $\mathbb{C}$ such that $\mathbb{C}^2 = 1$, and the Jost operators for $h < 0$ can be introduced through the relations

$$\begin{aligned} T_{-\varepsilon}\big(\mu, -h\big) &= \mathbb{C}\, T_\varepsilon\big(\mu, h\big)\, \mathbb{C} \\ Q_{-\varepsilon}\big(\mu, -h\big) &= \mathbb{C}\, Q_\varepsilon\big(\mu, h\big)\, \mathbb{C}\ . \end{aligned} \tag{4.20}$$

An important consequence of the Hermiticity, C- and T-invariance is that p-vacuum expectation values of Jost operators are expressed in terms of a single, real analytic function of $m \in (-1, 1)$:[2]

$$\begin{aligned} \langle\, p' \,|\, T_\pm(\mu, h) \,|\, p\,\rangle &= F(\pm m)\ \delta_{2p'-2p,m} \\ \langle\, p' \,|\, Q_\pm(\mu, h) \,|\, p\,\rangle &= F(\pm m)\ \delta_{2p-2p',-m}\ . \end{aligned} \tag{4.21}$$

As for the matrix elements (4.17), the C-invariance implies that $\mathcal{A}_\pm$ satisfy the relation (3.13) within the principal domain (3.14).

- ***Bilinear relation.*** As it follows from eq.(3.11), $\mathcal{A}_+(m_2, m_1\,|0) + \mathcal{A}_-(m_2, m_1\,|0) = 1$. Taking this into account along with the conjugation rule (4.19), the quantized version of the bilinear relation (4.11) is expected to be given by

$$\sum_{\varepsilon=\pm} Q_\varepsilon(\mu, h) T_\varepsilon(\mu, h) = 1|_{\mathcal{H}}\ . \tag{4.22}$$

- ***Lorentz invariance.*** Let us introduce the complex coordinate $z = x + \mathrm{i}\,y$ and the polar angle $\psi = \arg(\mathrm{i}\,z)$. The generator of infinitesimal Euclidean rotations coincides with $(-\mathrm{i}\,K)$, where K is the Lorentz boost generator. It is crucial for our analysis that the angular evolution of Jost operators turns out to be remarkably simple. Namely,

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\psi K}\ T_\varepsilon(\mu, h)\ \mathrm{e}^{+\mathrm{i}\psi K} &= T_\varepsilon\big(\mathrm{e}^{\mathrm{i}(1-g)\psi}\mu, \mathrm{e}^{\mathrm{i}\psi} h\big) \\ \mathrm{e}^{-\mathrm{i}\psi K}\ Q_\varepsilon(\mu, h)\ \mathrm{e}^{+\mathrm{i}\psi K} &= Q_\varepsilon\big(\mathrm{e}^{\mathrm{i}(1-g)\psi}\mu, \mathrm{e}^{\mathrm{i}\psi} h\big)\ . \end{aligned} \tag{4.23}$$

This follows from three observations; First, the exponential fields in (4.12) are chiral (holomorphic) $\big(\mathrm{e}^{\pm 2\mathrm{i}\phi(x,y)} \equiv \mathrm{e}^{\pm 2\mathrm{i}\phi}(z)\big)$ with the Lorentz spin g. Second, the exponential operator located at the origin is not affected by the Euclidean rotation: $\mathrm{e}^{-\mathrm{i}\psi K}\ \mathrm{e}^{\pm\mathrm{i}\phi(0,0)}\ \mathrm{e}^{+\mathrm{i}\psi K} = \mathrm{e}^{\pm\mathrm{i}\phi(0,0)}$. Finally, we have to accept that integrals containing combinations of the *holomorphic* fields are not changed by rotations of the integration contour in the limit $L \to \infty$.

The simple geometry of the Euclidean plane suggests that T- and Q- operators are related through the Euclidean rotation of angle π, combined with the C-conjugation:

$$Q_\varepsilon(\mu, h) = \mathbb{C}\, \mathrm{e}^{-\mathrm{i}\pi K}\ T_\varepsilon(\mu, h)\ \mathrm{e}^{\mathrm{i}\pi K}\, \mathbb{C}\ , \tag{4.24}$$

or, equivalently (see eqs.(4.20),(4.23))

$$Q_\varepsilon(\mu, h) = T_{-\varepsilon}\big(\mathrm{e}^{\mathrm{i}\pi(1-g)}\mu, h\big)\ . \tag{4.25}$$

[2] Notice that as it follows from (4.10), $\lim_{g\to 0} F(m) = \big(\frac{1+m}{2}\big)^{\frac{1}{2}}$.

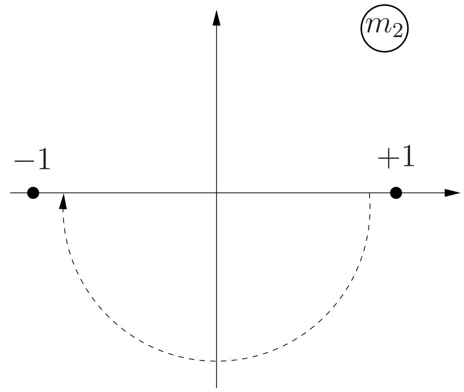

Figure 4: The contour of analytic continuation in eq.(4.26) as $m_1 \in (-1, 1)$ and $\alpha \in \mathbb{R}$.

It is worth to keep in mind that these formulae should be understood in a weak sense as relations for the analytic continuation of a certain class of matrix elements of Jost operators. The applicability of eqs.(4.23) (which is crucial for deriving (4.25)) requires that the contours of integration, involving in the construction of Jost operators, can be rotate freely within the Euclidean plane. This may be not the case for general matrix elements.

We expect that (4.25) can be applied for the matrix elements (4.17), and therefore the dimensionless amplitudes $\mathcal{A}_\pm$ are related by the analytic continuation with two-point functions containing T-operators only:

$$\langle\, p' \,|\, T_\mp(\mu_2, h_2)\, T_\pm(\mu_1, h_1)\,|\, p\rangle = \mathcal{A}_\pm\big(\mathrm{e}^{-\mathrm{i}\pi}\, m_2,\, m_1 \,|\, \alpha - \mathrm{i}\pi\big)\ \delta_{2p'-2p, m_1+m_2}\ . \tag{4.26}$$

Here the phase rotation $\mathrm{e}^{-\mathrm{i}\pi}$ means the analytic continuation along the clockwise half-circle of radius smaller then one (see Fig. 4) for $-1 < m_1 < 1$ and $\alpha \in \mathbb{R}$. Notice that the analytic continuation appearing in the r.h.s. of (4.26) does not involve any problem for the perturbative amplitudes (3.12), (3.13).[3] This supports rather sweeping assumptions that have been made in derivation of (4.26).

5 Algebra of Jost operators

At the moment it is not clear how to deal with the fidelities beyond the scope of perturbation theory for arbitrary values μ_i and h_i. However, in the case of $h_1 = h_2 \equiv h$ much can be said about the matrix elements

$$\langle\, p' \,|\, T_{\varepsilon_2}(\mu_2, h)\, T_{\varepsilon_1}(\mu_1, h)\,|\, p\,\rangle = F_{\varepsilon_2\varepsilon_1}(m_2, m_1)\ \ \delta_{2p'-2p, m_2+m_1}\ . \tag{5.1}$$

Notice that according to eq.(4.26)

$$\mathcal{A}_\pm\big(m_2,\, m_1 \,|\, \alpha\big)\big|_{h_1=h_2} = F_{\mp\pm}(\mathrm{e}^{\mathrm{i}\pi} m_2, m_1)\ , \tag{5.2}$$

[3] In this connection it deserves mentioning that eq.(3.13) should be understood as $\mathcal{A}_\varepsilon(m_2, m_1 \,|\, \alpha) = \mathcal{A}_\varepsilon(\mathrm{e}^{\pm\mathrm{i}\pi} m_2, \mathrm{e}^{\pm\mathrm{i}\pi} m_1 \,|\, \alpha)$, which is satisfied for any sign $\pm$.

and α is not an independent variable as $h_1 = h_2$. In this case α can be written as

$$\alpha = \frac{1}{1-g}\ \log\left(\frac{\lambda_2}{\lambda_1}\right)\ , \tag{5.3}$$

where λ_i $(i = 1, 2)$ stands for the dimensionless ratio μ_i/h^{1-g}, which is a certain function of m_i, i.e., $\lambda_i = \lambda(m_i)$. The corresponding inverse function $m = m(\lambda)$ is given by eq.(3.17). In the present discussion we will use both variables m_i and λ_i assuming that they are related through the formula (3.17). Since $h > 0$ is assumed to be fixed from now on, it makes sense to simplify the notation for the Jost operators:

$$T_\varepsilon(\mu, h) \equiv T_\varepsilon(m)\ , \qquad Q_\varepsilon(\mu, h) \equiv Q_\varepsilon(m)\ . \tag{5.4}$$

5.1 Commutation relations

Let us recall that $\phi(x, y)$ at $y = 0$ satisfies the commutation relation (2.10). Since the Euclidean time dependence of the chiral field $\phi(x, y)$ is very simple, we can translate (2.10) into the commutation relation at $x = 0$:

$$[\phi(0, y_2),\ \phi(0, y_1)] = \frac{\mathrm{i}}{2}\ \pi g\ \mathrm{sgn}(y_2 - y_1)\ . \tag{5.5}$$

According to ref. [22], if the matrix-valued operators $\boldsymbol{L}_i = \mathrm{e}^{\mathrm{i}\phi(0,y_2)}\boldsymbol{S}(y_2, y_1\,|\,\mu_i, h)$ $(i = 1, 2)$, where $\boldsymbol{S}$ is defined by eq.(4.12) with $y_2 > y_1$ and with μ_i are set to be y-independent constants, then the Yang-Baxter equation is satisfied in the form

$$\check{\boldsymbol{R}}\,\big(\,\boldsymbol{L}_1 \otimes 1\,\big)\,\big(\,1 \otimes \boldsymbol{L}_2\,\big) = \big(\,1 \otimes \boldsymbol{L}_2\,\big)\,\big(\,\boldsymbol{L}_1 \otimes 1\,\big)\,\check{\boldsymbol{R}}. \tag{5.6}$$

Nontrivial matrix elements of the 4×4 matrix $\check{\boldsymbol{R}}$ read explicitly

$$\check{R}^{++}_{++} = \check{R}^{--}_{--} = 1\,, \qquad \check{R}^{+-}_{+-} = \check{R}^{-+}_{-+} = \frac{\lambda_1^2 - \lambda_2^2}{\mathsf{q}\lambda_1^2 - \mathsf{q}^{-1}\lambda_2^2}\,, \qquad \check{R}^{-+}_{+-} = \check{R}^{+-}_{-+} = \frac{(\mathsf{q} - \mathsf{q}^{-1})\lambda_1\lambda_2}{\mathsf{q}\lambda_1^2 - \mathsf{q}^{-1}\lambda_2^2}\ , \tag{5.7}$$

where we use $\mathsf{q} = \mathrm{e}^{\mathrm{i}\pi g}$ and $\lambda_i = \mu_i/h^{1-g}$. Although a mathematically satisfactory construction of the Jost operators is absent, the arguments similar to those from ref. [21] suggest that the Jost operators satisfy the commutation relations

$$T_{\varepsilon_1}(m_1)T_{\varepsilon_2}(m_2) = \sum_{\varepsilon_1', \varepsilon_2' = \pm} R^{\varepsilon_1'\varepsilon_2'}_{\varepsilon_1\varepsilon_2}(m_1, m_2)\ T_{\varepsilon_2'}(m_2)T_{\varepsilon_1'}(m_1)\ , \tag{5.8}$$

where the R-matrix obeys the Yang-Baxter equation together with the "unitarity" and "crossing symmetry" relations

$$\sum_{\varepsilon_1', \varepsilon_2' = \pm} R^{\varepsilon_1'\varepsilon_2'}_{\varepsilon_1\varepsilon_2}(m_1, m_2) R^{\varepsilon_3\varepsilon_4}_{\varepsilon_2'\varepsilon_1'}(m_2, m_1) = \delta^{\varepsilon_4}_{\varepsilon_1}\delta^{\varepsilon_3}_{\varepsilon_2}\ , \tag{5.9}$$
$$\sum_{\varepsilon_1', \varepsilon_2', \varepsilon_3' = \pm} \delta^{\varepsilon_1' + \varepsilon_3', 0}\ R^{\varepsilon_1\varepsilon_2}_{\varepsilon_1'\varepsilon_2'}(\mathrm{e}^{\mathrm{i}\pi}m_1, m_2)\ R^{\varepsilon_3\varepsilon_2'}_{\varepsilon_3'\varepsilon_4}(m_1, m_2) = \delta^{\varepsilon_1 + \varepsilon_3, 0}\ \delta^{\varepsilon_2}_{\varepsilon_4}\ .$$

The formal rôle of the Yang-Baxter and unitarity relations is clear; the unitarity is required for self-consistency of (5.8), whereas the Yang-Baxter equation is the associativity constraint. In its turn, the crossing symmetry allows one to supplement algebraic relations (5.8) with an extra bilinear relation,

$$\sum_{\varepsilon=\pm} T_{-\varepsilon}(\mathrm{e}^{\mathrm{i}\pi}m)\, T_{\varepsilon}(m) = 1 \ , \tag{5.10}$$

which follows from eqs. (4.22) and (4.25).

The R-matrix in the form

$$R^{\varepsilon_1'\varepsilon_2'}_{\varepsilon_1\varepsilon_2}(m_1,m_2) = R(m_1,m_2)\ \check{R}^{\varepsilon_1'\varepsilon_2'}_{\varepsilon_1\varepsilon_2} \ , \tag{5.11}$$

meet all the necessary requirements, if the normalization factor satisfies the conditions

$$\begin{aligned} &R(m_1,m_2)\, R(m_2,m_1) = 1 \\ &R(m_1,m_2)\, R(\mathrm{e}^{\mathrm{i}\pi}m_1,m_2) = \frac{\mathtt{q}\lambda_1^2 - \mathtt{q}^{-1}\lambda_2^2}{\lambda_1^2-\lambda_2^2} \ . \end{aligned} \tag{5.12}$$

(recall that we use the convention $\lambda_i = \lambda(m_i)$). Some extra conditions are imposed by the global symmetries. Namely, the Hermiticity and T-invariance require that

$$R(m_1,m_2)R^*(m_1,m_2) = 1 \qquad \text{for} \quad 0 < m_{1,2} < 1 \ , \tag{5.13}$$

whereas the C-symmetry yields the relation

$$R\big(\mathrm{e}^{\pm\mathrm{i}\pi}m_1, \mathrm{e}^{\pm\mathrm{i}\pi}m_2\big) = R(m_1,m_2) \ . \tag{5.14}$$

5.2 Normalization factor $R(m_1, m_2)$

The algebra of the Jost operators together with the global symmetry relations and normalizations conditions lead to a system of functional equations imposed on the two-point function $F_{\varepsilon_2\varepsilon_1}(m_2,m_1)$ (5.1). Assuming that $0 < m_{1,2} < 1$, the system reads as follows:

$$F_{\varepsilon_1\varepsilon_2}(m_1,m_2) = \sum_{\varepsilon_1',\varepsilon_2'=\pm} R^{\varepsilon_1'\varepsilon_2'}_{\varepsilon_1\varepsilon_2}(m_1,m_2) F_{\varepsilon_2'\varepsilon_1'}(m_2,m_1) \tag{5.15}$$

and

$$F_{\varepsilon_2\varepsilon_1}(\mathrm{e}^{\mathrm{i}\pi}m_2,m_1) = F_{-\varepsilon_1,-\varepsilon_2}(\mathrm{e}^{\mathrm{i}\pi}m_1,m_2) = F_{-\varepsilon_2,-\varepsilon_1}(m_2,\mathrm{e}^{-\mathrm{i}\pi}m_1) = \big(F_{\varepsilon_2\varepsilon_1}(\mathrm{e}^{\mathrm{i}\pi}m_2,m_1)\big)^* \tag{5.16}$$

and

$$F_{-\varepsilon\varepsilon}(\mathrm{e}^{\mathrm{i}\pi}m, m\,|\,0\,) = \frac{1}{2}\ (1+\varepsilon\, m) \ . \tag{5.17}$$

Using the perturbative results (3.12), one can check that all the conditions for $F_{\mp\pm}$ are satisfied at the first perturbative order, provided the R-matrix has the form (5.11) with

$$\log R(m_1,m_2) = \frac{\mathrm{i}}{2}\,\pi g\ \ \frac{m_2-m_1}{m_2+m_1}\ \ (1+m_2m_1) + O(g^2) \ . \tag{5.18}$$

Without additional analytical assumptions the set (5.12)-(5.14) and (5.18) does not unambiguously define $R(m_1, m_2)$. However, the first perturbative correction allows one to make a conjecture about the exact normalization factor. Namely, a simple calculation shows that, at least at the first perturbative order, the normalization factor obeys the condition

$$\left(\lambda_2 \frac{\partial}{\partial\lambda_2} + \lambda_1 \frac{\partial}{\partial\lambda_1}\right) \log R(m_1, m_2) = \frac{\mathrm{i}}{2}\, \pi g \left(\lambda_2 \frac{\partial}{\partial\lambda_2} - \lambda_1 \frac{\partial}{\partial\lambda_1}\right) m_1 m_2 \ . \tag{5.19}$$

Here $m_i \equiv m(\lambda_i)$ $(i = 1, 2)$. If we accept (5.19) for any $0 < g < 1$, the reconstruction of $\log R(m_1, m_2)$ requires a minimum amount of additional analytical assumptions. Indeed, consider the limit when $h \to 0$, keeping the Kondo temperatures $E_1^\star$ and $E_2^\star$ fixed. In this case both m_1 and m_2 turns to be zero, but their ratio remains finite. Then

$$\lim_{\mathfrak{h}\to 0} R\big(\mathfrak{h}\mathrm{e}^{\alpha_1}, \mathfrak{h}\mathrm{e}^{\alpha_2}\big) = R_0(\alpha_1 - \alpha_2) \ , \tag{5.20}$$

where $\alpha_1 - \alpha_2 = \log(E_2^\star/E_1^\star)$, and as it follows from eqs.(5.12),

$$R_0(\alpha) R_0(-\alpha) = 1 \ , \qquad R_0(\alpha + \mathrm{i}\pi) R_0(\alpha) = -\frac{\sinh(1-g)(\alpha + \mathrm{i}\pi)}{\sinh(1-g)\alpha} \ . \tag{5.21}$$

(Notice that for unrelated h_1 and h_2 the variable α coincides with the one defined by eq.(3.10).) The "minimal" solution of (5.21) (i.e., such that $\log R_0(\alpha)$ is analytic in the strip $0 \leq \Im m(\alpha) \leq \pi$ and bounded at $\alpha \to +\infty$) has the form

$$R_0(\alpha) = \exp\left(-\mathrm{i} \int_0^\infty \frac{\mathrm{d}\omega}{\omega}\, \frac{\sin(\alpha\omega)}{\cosh\frac{\pi\omega}{2}}\, \frac{\sinh\frac{\pi g \omega}{2(1-g)}}{\sinh\frac{\pi\omega}{2(1-g)}}\right) \ . \tag{5.22}$$

Eq.(5.19), as a linear partial differential equation subject of the asymptotic condition (5.20), can be easily solved by means of the Fourier transform. Using (3.17) one finds

$$R(m_1, m_2) = \exp\left(\frac{g}{8\pi\mathrm{i}} \iint_{-\infty}^{+\infty} \mathrm{d}\omega_1 \mathrm{d}\omega_2\ \lambda_1^{\frac{\mathrm{i}\omega_1}{1-g}} \lambda_2^{\frac{\mathrm{i}\omega_2}{1-g}}\ \frac{M(\omega_1)\, M(\omega_2)}{(\omega_1 + \mathrm{i}0)(\omega_2 + \mathrm{i}0)}\ \frac{\omega_2 - \omega_1}{\omega_2 + \omega_1 - \mathrm{i}0}\right) . \tag{5.23}$$

The function (5.23) meets the required conditions (5.12)-(5.14). It also possesses a small-g series expansion

$$\log R(m_1, m_2) = \mathrm{i}\pi \sum_{n=1}^{\infty} g^n\ r_n(m_1, m_2) \ , \tag{5.24}$$

with

$$\begin{aligned} r_1 &= \frac{1}{2}\, \frac{m_2 - m_1}{m_1 + m_2}\, (1 + m_1\, m_2) \\ r_2 &= \frac{m_1 m_2 (1 - m_1^2)(1 - m_2^2)}{(m_1^2 - m_2^2)^2} \left(m_1^2 - m_2^2 + 2 m_1 m_2 \log\left(\frac{m_2}{m_1}\right)\right) \\ r_3 &= \frac{m_1 m_2 (1 - m_1^2)(1 - m_2^2)}{(m_1^2 - m_2^2)^3} \bigg((m_1^2 - m_2^2)^2\ (1 + m_1^2 + m_2^2 - 3 m_1 m_2) - \\ &\frac{\pi^2}{12}\, (m_1 - m_2)^4\ (m_1^2 + m_2^2 + m_1 m_2 - 1) + 2\ m_1 m_2 (2 m_1^2 m_2^2 - m_1^2 - m_2^2)\ \log^2\left(\frac{m_2}{m_1}\right)\bigg), \mathit{etc.} \end{aligned} \tag{5.25}$$

It would be valuable to confirm these higher-order corrections using the renormalized perturbation theory. More importantly, the origin of (5.19) (if it is true, of course) needs to be clarified. It may be useful to note here that the relation (5.19) can be equivalently written as

$$\Big(\lambda_2 \frac{\partial}{\partial\lambda_2} + \lambda_1 \frac{\partial}{\partial\lambda_1}\Big) \log\tilde{R}(m_1,m_2) = \mathrm{i}\pi\, \Big(\lambda_2 \frac{\partial}{\partial\lambda_2} - \lambda_1 \frac{\partial}{\partial\lambda_1}\Big)\, d_{21}(-m_2,m_1)\ , \tag{5.26}$$

where $\tilde{R}(m_1,m_2) = \mathrm{e}^{\frac{\mathrm{i}\pi g}{4}(m_2^2-m_1^2)}\, R(m_1,m_2)$ and $d_{21} = \frac{g}{4}\,(m_2-m_1)^2$ is the orthogonality exponent.

5.3 Fidelities $A_{21}(\sigma_{0,3})$ for $h=0$

As $h\to 0$, the system of functional equations (5.15)-(5.17) is simplified dramatically. In this case

$$\lim_{\mathfrak{h}\to 0^+} F_{\pm\mp}\big(\mathfrak{h}\mathrm{e}^{\alpha_2},\mathfrak{h}\mathrm{e}^{\alpha_1}\big) = f(\alpha_1-\alpha_2) \tag{5.27}$$

and f satisfy the equations:

$$f(\mathrm{i}\pi+\alpha) = f(\mathrm{i}\pi-\alpha)\ , \qquad \frac{f(-\alpha)}{f(\alpha)} = \frac{\sinh\frac{1-g}{2}\,(\mathrm{i}\pi+\alpha)}{\sinh\frac{1-g}{2}\,(\mathrm{i}\pi-\alpha)}\ R_0(\alpha)\ . \tag{5.28}$$

Let us assume now that $\log f$ is analytic in the strip $0<\Im m(\alpha)\le\pi$ and grows slower then any exponential of α at infinity.[4] With these analytical assumptions, the functional relations (5.28) define f up to a constant multiplicative factor. Then the normalization condition (5.17) fixes f unambiguously. Finally, using relations (5.2), we derive

$$\begin{aligned} A_{21}(\sigma_0)|_{h_1=h_2=0} &= 2\,f(\alpha-\mathrm{i}\pi) = u(\alpha)\,\frac{\sinh(1-g)\,\frac{\alpha}{2}}{(1-g)\sinh\frac{\alpha}{2}} \\ A_{21}(\sigma_3)|_{h_1=h_2=0} &= 0\ , \end{aligned} \tag{5.29}$$

where

$$u(\alpha) = \exp\bigg(\int_0^\infty \frac{\mathrm{d}\omega}{\omega}\,\frac{\sin^2(\alpha\omega/2)}{\sinh\pi\omega\cosh\frac{\pi\omega}{2}}\ \frac{\sinh\frac{\pi g\omega}{2(1-g)}}{\sinh\frac{\pi\omega}{2(1-g)}}\bigg)\ . \tag{5.30}$$

A numerical verification of the prediction (5.29) was performed in ref. [23].

6 Concluding remarks

Although the problem of finding fidelities $A_{21}(\sigma_s)$ for non-vanishing h remains unsolved, it seems useful to discuss its generalization.

[4] These assumptions can be supported by the perturbative results (3.12). Also note that as $g=0$ and $h=0$, the natural domain of analyticity of the classical Jost functions $T^{(c)}_{\pm}$ (4.10), as functions of the complex variable $J=\mathtt{k}|_{h=0}$, is the lower half plane $-\pi\le\arg(J)\le 0$ which corresponds to the strip $0\le\Im m(\alpha)\le\pi$.

The boundary operators σ_s are particular representatives of the family of "primary" boundary fields

$$\sigma_s^{(a)} = \mathrm{e}^{\mathrm{i}a\Phi_B}\ \sigma_s \qquad (s=0,3,\pm) \tag{6.1}$$

(here $\sigma_\pm = \sigma_1 \pm \mathrm{i}\sigma_2$). Recall that the Hamiltonian $\boldsymbol{H}$(2.11) relates to the renormalized Hamiltonian $\boldsymbol{H}_\phi$ (2.15) by means of the canonical transformation $\boldsymbol{H}_\phi = \boldsymbol{U}^\dagger \boldsymbol{H}\boldsymbol{U}$ with $\boldsymbol{U} = \exp\big(\mathrm{i}\sigma_3\phi(0)\big)$ and $\Phi_B(0) = 2\phi(0)$. For this reason the scale dimension of $\sigma_\pm \mathrm{e}^{\mathrm{i}a\Phi_B}$ is given by $D_\pm(a) = g(a\mp 1)^2$ whereas the scale dimension of $\sigma_{0,3}\mathrm{e}^{\mathrm{i}a\Phi_B}$ equals to $D_{0,3}(a) = ga^2$. An overall normalization of the primary boundary fields can be chosen to satisfy the normalization condition at $y\to 0^+$,

$$\begin{aligned}
&\sigma_s^{(-a)}(y)\sigma_s^{(a)}(0) \to \sigma_0\ y^{-2ga^2} + \ldots && (s=0,3)\\
&\sigma_{-s}^{(-a)}(y)\sigma_s^{(a)}(0) \to \frac{1}{2}\ (\sigma_0 - s\,\sigma_3)\ y^{-2g(a-s)^2} + \ldots && (s=\pm)\ ,
\end{aligned} \tag{6.2}$$

along with the hermitian conjugation

$$\big[\sigma_s^{(a)}(y)\big]^\dagger = \begin{cases} \sigma_s^{(-a)}(-y) & \text{for} \quad s=0,3 \\ \sigma_{-s}^{(-a)}(-y) & \text{for} \quad s=\pm \end{cases}\ . \tag{6.3}$$

Repeating the construction from Section 2.3, one can introduce the normalized partition function $\bar{\mathcal{Z}}_{21}\big(\mathcal{O}|y\big)$ (2.18) for the operators $\mathcal{O} = \sigma_s^{(a)}$. As $y\to 0^+$, the effect of the boundary interaction becomes negligible within the interval $(0,y)$ and the pair of hermitian conjugated operators at the ends of the segment can be substituted by a local insertion defined by the leading term of the operator product expansions (6.2). This yields the leading behavior at $y\to 0^+$:

$$\bar{\mathcal{Z}}_{21}\big(\sigma_s^{(a)}|y\big) = \begin{cases} y^{-2ga^2}\ \big(1+o(1)\big) & \text{for} \quad s=0,3 \\ \frac{1}{2}\,(1-sm_1)\ y^{-2g(a-s)^2}\ \big(1+o(1)\big) & \text{for} \quad s=\pm \end{cases}\ . \tag{6.4}$$

The spin degrees of freedom are freezing out at large distances so that the large-y asymptotic of $\bar{\mathcal{Z}}_{21}\big(\sigma_s^{(a)}|y\big)$ has a similar form as eq.(2.23),

$$\bar{\mathcal{Z}}_{21}\big(\sigma_s^{(a)}|y\big) = \big|A_{21}\big(\sigma_s^{(a)}\big)\big|^2\ \ y^{-2d_{21}(a)}\ \mathrm{e}^{-y\Delta E_{21}}\ \big(1+o(1)\big) \qquad \text{as} \quad y\to+\infty\ . \tag{6.5}$$

The IR exponent $d_{21}(a)$ can be found using the mean field approximation from Section 3.1

$$d_{21}(a) = \frac{g}{4}\ (m_2 - m_1 - 2a)^2\ . \tag{6.6}$$

Just as in the case $a=0$, this formula is expected to be an exact result. This supposition is supported by a perturbative calculation given in the appendix. As before, $A_{21}\big(\sigma_s^{(a)}\big)$ in (6.5) can be thought as scaling functions depending on the m_1, m_2, and the pair of Kondo temperatures $E_1^\star$, $E_2^\star$. In the context of the spin-boson model (1.4), the fidelity $A_{21}\big(\sigma_s^{(a)}\big)$ represents the universal part of the vacuum-vacuum matrix element $\langle\, \Omega_2\,|\,\big[\sigma_s^{(a)}\big]_{\text{bare}}\,|\,\Omega_1\rangle$ of the "bare" operator (1.6), which requires both UV and IR regularizations. If the integration in (1.4) and (1.6) is restricted to the finite domain $L^{-1} < k < \Lambda$, then

$$\langle\, \Omega_2\,|\,\big[\sigma_s^{(a)}\big]_{\text{bare}}\,|\,\Omega_1\rangle \propto A_{21}\big(\sigma_s^{(a)}\big)\ L^{-d_{21}(a)}\ \Lambda^{-D_s(a)}\ . \tag{6.7}$$

In the case $h_1 = h_2$, the calculation of $A_{21}\big(\sigma_s^{(a)}\big)$ can be reduced to finding p-vacuum matrix elements generalizing (5.1):

$$\langle\, p' \,|\, \mathrm{e}^{2\mathrm{i}a\phi(0)}\, T_{\varepsilon_2}(m_2)\, T_{\varepsilon_1}(m_1)\,|\, p\,\rangle = F^{(a)}_{\varepsilon_2\varepsilon_1}(m_2, m_1)\ \ \delta_{2p'-2p,2a+m_2+m_1}\ . \tag{6.8}$$

The bosonization approach developed in ref. [21] allows one to study the case $h_1 = h_2 \to 0^+$ and extend the result from Section 5.3. It turns out that, for the primary field $\sigma_+^{(a)}$ with $0 \le a \le 1$, the fidelity is simply expressed in terms of the function $u(\alpha)$ (5.30):

$$A_{21}\big(\sigma_+^{(a)}\big)\big|_{h_1=h_2\to 0^+} = \mathcal{S}_+(a)\ \ u(\alpha)\ \big(E_1^\star E_2^\star\big)^{g(\frac{1}{2}-a)}\ , \tag{6.9}$$

where $\alpha = \log(E_2^\star/E_1^\star)$. In the case of $\sigma_{0,3}^{(a)}$, the corresponding fidelities are given by

$$A_{21}\big(\sigma_s^{(a)}\big)\big|_{h_1=h_2\to 0^+} = \mathcal{S}_0(a)\ \big(\,\mathrm{e}^{\mathrm{i}\pi g a}\ \mathcal{A}^{(a)}(\alpha) + (-1)^s\ \mathrm{e}^{-\mathrm{i}\pi g a}\ \mathcal{A}^{(-a)}(\alpha)\,\big) \qquad (s = 0, 3)\ , \tag{6.10}$$

where

$$\mathcal{A}^{(a)}(\alpha) \ \ = \ \ u(\alpha) \int_{-\infty}^{+\infty} \frac{\mathrm{d}\gamma}{2\pi}\ \frac{\mathrm{e}^{ga(2\gamma-\alpha)}}{\cosh\gamma}\ \frac{u(\gamma+\frac{\mathrm{i}\pi}{2})u(\gamma-\frac{\mathrm{i}\pi}{2})}{u(\gamma-\alpha+\frac{\mathrm{i}\pi}{2})u(\gamma-\alpha-\frac{\mathrm{i}\pi}{2})}\ . \tag{6.11}$$

Note that $\mathcal{S}_+(a)$ and $\mathcal{S}_0(a)$ in eqs.(6.9), (6.10) stand for some dimensionless functions of a and the coupling g, which remained undetermined except their values at $a = 0$:

$$\mathcal{S}_+(0) = -\frac{1}{\pi g}\ \tan\Big(\frac{\pi g}{2(1-g)}\Big)\ \ \Gamma(1-g)\left(\frac{g\sqrt{\pi}\,\Gamma(\frac{g}{2(1-g)})}{2\Gamma(\frac{1}{2}+\frac{g}{2(1-g)})}\right)^{1-g}\ , \qquad S_0(0) = 1\ . \tag{6.12}$$

Finally, one can consider the fidelities for scaling fields of the form

$$P_N\big(\partial_y\Phi_B, \partial_y^2\Phi_B, \ldots\big)\ \mathrm{e}^{\mathrm{i}a\Phi_B}\ \sigma_s\ ,$$

where $P_N(\ldots)$ stands for a differential polynomial of order N built from the boundary field $\partial_y\Phi_B$. In this most general setting, the problem is related to finding the multipoint amplitudes

$$\langle\, p' \,|\, \mathrm{e}^{2\mathrm{i}a\phi(0)}\, T_{\varepsilon_n}(m_n)\ \cdots T_{\varepsilon_1}(m_1)\,|\, p\,\rangle = F^{(a)}_{\varepsilon_n\ldots\varepsilon_1}(m_n, \ldots, m_1)\ \delta_{2p'-2p,-2a+m_n+\cdots+m_1}\ . \tag{6.13}$$

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Figure 5: The Anderson-Yuval [9] one-dimensional gas with an external field h and external charges $\pm a$ at the ends of segment. The arrow shows the direction of the external force exerted on the "positive" charges.

A Appendix: Conformal Perturbation Theory

Here we consider the large y asymptotic of the normalized partition function $\bar{\mathcal{Z}}_{21}\big(\sigma_0^{(a)}|y\big)$ for $E_1^\star = 0,\ h_1 > 0$, or equivalently, $m_1 = 1$. Note that in this case

$$\bar{\mathcal{Z}}_{21}\big(\sigma_3^{(a)}|y\big) = \bar{\mathcal{Z}}_{21}\big(\sigma_0^{(a)}|y\big)\ , \qquad \bar{\mathcal{Z}}_{21}\big(\sigma_-^{(a)}|y\big) = 0\ . \tag{A.1}$$

Using the renormalized Hamiltonian (2.15), where μ is renormalized coupling corresponding to the Kondo temperature $E^\star \equiv E_2^\star$ and $h \equiv h_2$, the ratio $\bar{\mathcal{Z}}_{21}\big(\sigma_0^{(a)}|y\big) = \mathcal{Z}_{21}\big(\sigma_0^{(a)}|y\big)/\mathcal{Z}_{11}$ can be represented in the form of the grand canonical partition function of the Anderson-Yuval one-dimensional gas of alternating charges on the finite interval,

$$\begin{aligned}\bar{\mathcal{Z}}_{21}(\sigma_0^{(a)}|y)\big|_{E_1^\star=0} &= \sum_{n=0}^{\infty}\mu^{2n}\int_0^y \mathrm{d}y_{2n}\int_0^{y_{2n}}\mathrm{d}y_{2n-1}\cdots\int_0^{y_2}\mathrm{d}y_1\prod_{l=1}^{n}\mathrm{e}^{2h(y_{2l-1}-y_{2l})} \\ &\times\ \big\langle\, p\,\big|\,\mathrm{e}^{-2\mathrm{i}a\phi(\tau)}\ \mathrm{e}^{-2\mathrm{i}\phi(y_{2n})}\mathrm{e}^{+2\mathrm{i}\phi(y_{2n-1})}\cdots\mathrm{e}^{-2\mathrm{i}\phi(y_2)}\,\mathrm{e}^{2\mathrm{i}\phi(y_1)}\,\mathrm{e}^{+2\mathrm{i}a\phi(0)}\big|\,p\,\big\rangle\ ,\end{aligned}\tag{A.2}$$

where, in fact, the matrix element does not depend on the choice of p-vacuum,

$$\big\langle\, p\,\big|\,\mathrm{e}^{2\mathrm{i}a_n\phi}(y_n)\cdots\mathrm{e}^{2\mathrm{i}a_1\phi}(y_1)\,\big|\,p\,\big\rangle = \delta_{a_1+\cdots+a_n,0}\ \prod_{i>j}(y_i-y_j)^{2g\,a_ia_j}\ . \tag{A.3}$$

The parameter h plays the rôle of the external electric field whereas the exponentials $\mathrm{e}^{\pm 2\mathrm{i}a\phi}$ can be interpreted as creators of two external charges at the ends of the interval (see Fig. 5). Then one has

$$\big|A(\sigma_0^{(a)})\big|^2_{E_1^\star=0} = h^{2ga^2-2\tilde{d}_{21}}\lim_{\ell=hy\to+\infty}\ell^{2\tilde{d}_{21}-2ga^2}\,\mathrm{e}^{\ell(1-e_0(\lambda))}\,\bigg(1+\sum_{n=0}^{\infty}\lambda^{2n}\ q_n(\ell)\bigg)\ . \tag{A.4}$$

Here $\tilde{d}_{21}$ is some (a priori unknown) orthogonality exponent,

$$e_0(\lambda)\equiv -E_0/h>0\ ,\qquad \lambda\equiv\mu/h^{1-g}\ , \tag{A.5}$$

and

$$\begin{aligned}q_n(\ell) &= \ell^{2n(1-g)}\int_0^1\mathrm{d}u_{2n}\int_0^{u_{2n}}\mathrm{d}u_{2n-1}\cdots\int_0^{u_2}\mathrm{d}u_1\,\prod_{i=1}^{n}\mathrm{e}^{2\ell(u_{2i-1}-u_{2i})}\,\bigg[\frac{(1-u_{2i})u_{2i-1}}{(1-u_{2i-1})u_{2i}}\bigg]^{2ag} \\ &\times\ \prod_{i,j=1}^{n}|u_{2i}-u_{2j-1}|^{-2g}\prod_{1\le i<j\le n}\big|(u_{2j}-u_{2i})(u_{2j-1}-u_{2i-1})\big|^{2g}\ .\end{aligned}\tag{A.6}$$

Consider the first nontrivial coefficient $q_1(\ell)$. One can show that as $\ell\to+\infty$,

$$q_1(\ell)=2^{2g-1}\,\Gamma(1-2g)\ \ell-2^{2g-2}\,\Gamma(2-2g)\ \Big(1+4ag\ \log\big(\ell\mathrm{e}^{C}\big)\Big)+o(1)\ , \tag{A.7}$$

where $C=1+\log(2)-\gamma_E-\psi(2-2g)-\psi(1+2ag)$ and $\psi(z)\equiv\partial_z\log\Gamma(z)$. Substituting (A.7) in (A.4) we observe that the term $\propto\lambda^2\ell$ is canceled out provided that $e_0(\lambda)=1+2^{2g-1}\,\Gamma(1-2g)\ \lambda^2+O(\lambda^4)$. Taking into account that $m(\lambda)=e_0-(1-g)\lambda\,\frac{\partial e_0}{\partial\lambda}$, i.e.,

$$m(\lambda)=1-2^{2g-1}\,\Gamma(2-2g)\ \lambda^2+O(\lambda^4)\ , \tag{A.8}$$

one finds that the cancellation of the divergent term $\propto\lambda^2\log\ell$ requires that

$$\tilde d_{21}=ga^2+ga\,(1-m)+O\big((1-m)^2\big)\ , \tag{A.9}$$

which is in agreement with (6.6) specialized for $m_1=1$, $m_2=m$. The above calculation suggests that

$$\big|A_{21}(\sigma_0^{(a)})\big|_{E_1^\star=0}=h^{-ga(1-m)-\frac{g}{4}(1-m)^2}\ A(m,ag)\ , \tag{A.10}$$

where

$$A(m,l)=\sqrt{\frac{1+m}{2}}\ \Big(1+\sum_{n=1}^{\infty}\mathfrak{a}_n(l)\ \ (1-m)^n\Big)\ , \tag{A.11}$$

and the first expansion coefficient reads explicitly

$$\mathfrak{a}_1(l)=-l\ \big(1+\log(2)-\gamma_E-\psi(2-2g)-\psi(1+2l)\big)\ . \tag{A.12}$$

The function (A.10) is expected to have a finite limit as $h\to0^+$:

$$\lim_{h\to0^+}\big|A_{21}(\sigma_0^{(a)})\big|_{E_1^\star=0}=\big(E^\star\big)^{-g(a+\frac{1}{4})}\ \mathcal{A}_0^{(a)}\ . \tag{A.13}$$

The dimensionless amplitude $\mathcal{A}_0^{(a)}$ is the universal part of the partition function $\mathcal{Z}^{(a)}$ of the system depicted in Fig. 6,

$$\mathcal{Z}^{(a)}/\mathcal{Z}_{\rm free}\sim\mathcal{A}_0^{(a)}\ (c_1\varepsilon E^\star)^{a^2}\ (c_2LE^\star)^{-g(a+\frac{1}{2})^2}\ . \tag{A.14}$$

Here $\mathcal{Z}_{\rm free}$ is the partition function of the Gaussian CFT with free BC, ε is the lattice spacing, and a-independent, non-universal coefficients c_i depend on details of the IR and UV regularizations.

It deserves to be mentioned here that $\big|A_{21}(\sigma_0^{(a)})\big|_{E_1^\star=0}$ has a finite limit as $g\to0$ and $l=ag$ is kept fixed. In this case, as it follows from (A.2), (A.3),

$$\lim_{\substack{g\to0\\ l=ag-\text{fixed}}}\ y^{2ga^2}\,\bar{\mathcal{Z}}_{21}\big(\sigma_0^{(a)}|y\big)\big|_{J_1=0}=\langle\uparrow|\mathcal{T}_\tau\exp\Big[J\int_0^y\mathrm{d}\tau\big(\mathrm{e}^{2\mathrm{i}\phi_c(-\mathrm{i}\tau)}\,\sigma_-+\mathrm{e}^{-2\mathrm{i}\phi_c(-\mathrm{i}\tau)}\,\sigma_+\big)\Big]|\uparrow\rangle \tag{A.15}$$

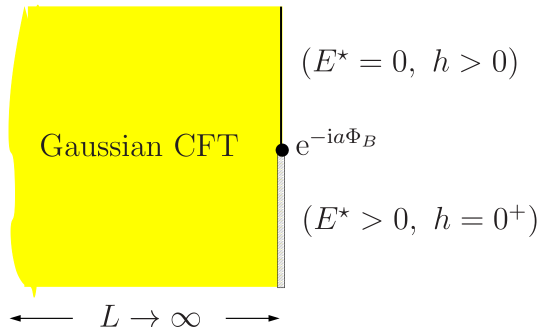

Figure 6: The dimensionless amplitude $\mathcal{A}_0^{(a)}$ (A.13) is the universal part of the partition function of the statistical system schematically shown in this figure.

where $\mathrm{i}\,\phi_c(-\mathrm{i}\tau) = h\tau + l\ \log\left(\frac{\tau}{y-\tau}\right)$. In the presence of singularities at $\tau = 0,\, y$, the analysis performed in Section 4.1 requires some modification. One can show that $A(m,l)$ at $g = 0$ coincides with the connection coefficient

$$T_+^{(c)}(m,l) = \sqrt{\frac{1+m}{2}}\ \lim_{\tau\to 0^-} (-\tau)^l\ \Psi_{\mathtt{k}}(-\mathrm{i}\tau)\ , \tag{A.16}$$

where $\Psi_{\mathtt{k}}(-\mathrm{i}\tau)$ is the Jost solution of the Sturm-Liouville equation

$$-\partial_\tau^2\Psi + \left(1 - \frac{2\mathtt{k}lm}{\tau} + \frac{l(l+1)}{\tau^2}\right)\Psi = 0\ , \tag{A.17}$$

satisfying the asymptotic condition

$$\Psi_{\mathtt{k}}(-\mathrm{i}\tau) \to (-\tau)^{-lm}\ \mathrm{e}^{\mathtt{k}\tau} \qquad \text{as} \qquad \tau \to -\infty \tag{A.18}$$

(recall that we use the notations $\mathtt{k} = \sqrt{J^2+h^2} > 0$ and $m = h/\mathtt{k}$). The Jost solution is expressed in terms of the confluent hypergeometric function:

$$\Psi_{\mathtt{k}}(-\mathrm{i}\tau) = \mathtt{k}^{lm}\ (-\mathtt{k}\tau)^{1+l}\ \mathrm{e}^{\mathtt{k}\tau}\ \ 2^{1+l(1+m)}\ U\big(1+(1+m)l\,,\, 2(1+l)\,,\, -2\mathtt{k}\tau\big)\ . \tag{A.19}$$

This yields the result

$$T_+^{(c)}(m,l) = \sqrt{\frac{1+m}{2}}\ \ m^{l(1-m)}\ \ \frac{2^{l(m-1)}\Gamma(1+2l)}{\Gamma(1+(1+m)l)}\ . \tag{A.20}$$

Notice that an expansion of $T_+^{(c)}(m,l)$ at $m=1$ turns out to be consistent with the classical limit of (A.10)-(A.12). Thus we see that $\big|A_{21}(\sigma_0^{(a)})\big|_{\substack{E_1^\star=0\\ g=0}} = h^{-l}\,T_+^{(c)}(m,l)$, and it is an analytic function for $\Re e(l) > -\frac{1}{2}$. There are two other points which deserve notice, such as the bilinear relation

$$T_+^{(c)}(\mathrm{e}^{\mathrm{i}\pi}m,l)\,T_-^{(c)}(m,l) + T_-^{(c)}(\mathrm{e}^{\mathrm{i}\pi}m,l)\,T_+^{(c)}(m,l) = 1\ , \tag{A.21}$$

where

$$T_-^{(c)}(m,l) = \sqrt{\frac{1-m}{2}}\ \ m^{-l(1+m)}\ \ \frac{2^{l(m+1)}\Gamma(1-2l)}{\Gamma(1-(1-m)l)}\ , \tag{A.22}$$

(to be compared with eq.(5.10)), and the behavior of $T_+^{(c)}(l,m)$ as $m \to 0^+$:

$$T_+^{(c)}(m,l) \to m^l \; 2^l \; \frac{\Gamma(\frac{1}{2}+l)}{\sqrt{2\pi}} \; . \tag{A.23}$$

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